

**Report on my Research on some
Regularity Questions
Regarding Multivalued/ Q-valued
Functions**

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Zusammenfassung:

Wir befassen uns mit einigen Fragestellungen zur Regularität von Almgren's Q -valued functions.

Insbesondere betrachten wir Regularität am Rand für Minimierer der Dirichlet-Energie und die Hölder-Stetigkeit von Energie minimierenden Q -valued harmonic maps.

Q -valued functions sind mehrwertige Funktionen mit genau Q Werten, wobei Multiplizitäten berücksichtigt werden. Diese Funktionen wurden von F. Almgren 1983 eingeführt. Wir können zeigen dass Minimierer der Dirichlet-Energie unter gewissen Voraussetzungen an die Regularität der Randwerte und des Gebietes, Hölder-stetig sind bis zum Rand.

Energie minimierende Q -valued functions die nur Werte in einer Riemannschen Mannigfaltigkeit annehmen sind zumindest Hölder-stetig auf einer großen Teilmenge im Inneren ihres Gebietes.

Des Weiteren geben wir Beispiele für holomorphe Funktionen die Nullstellen unendlicher Ordnung an Randpunkten besitzen. Diese können als Einladungen dienen Randregularität näher zu betrachten. Die präsentierten holomorphen Funktionen haben die überraschende Eigenschaft, dass ihre Nullstellenmenge zum Rand hin sehr groß wird. Trotz allem sind sie selbst nicht konstant Null.

Summary:

We address various regularity questions concerning Almgren's Q -valued functions.

In particular boundary regularity for Q -valued Dirichlet minimizers and interior Hölder regularity for Q -valued harmonic maps is considered.

Q -valued functions are multiple valued functions taking exactly Q values, counting multiplicity introduced by F. Almgren 1983. We are able to show that such a functions minimizing the energy of its slope behave Hölder continuous up to the boundary if the boundary data and domain are sufficient regular. If the target is restricted to be a Riemannian manifold such an energy minimising functions is at least Hölder continuous on a large portion of the interior of its domain.

Furthermore we present examples of holomorphic functions vanishing to infinite order at boundary points as an invitation to discuss boundary regularity. Surprisingly functions of this specific class can vanish on a large set towards the boundary still not being constant zero.

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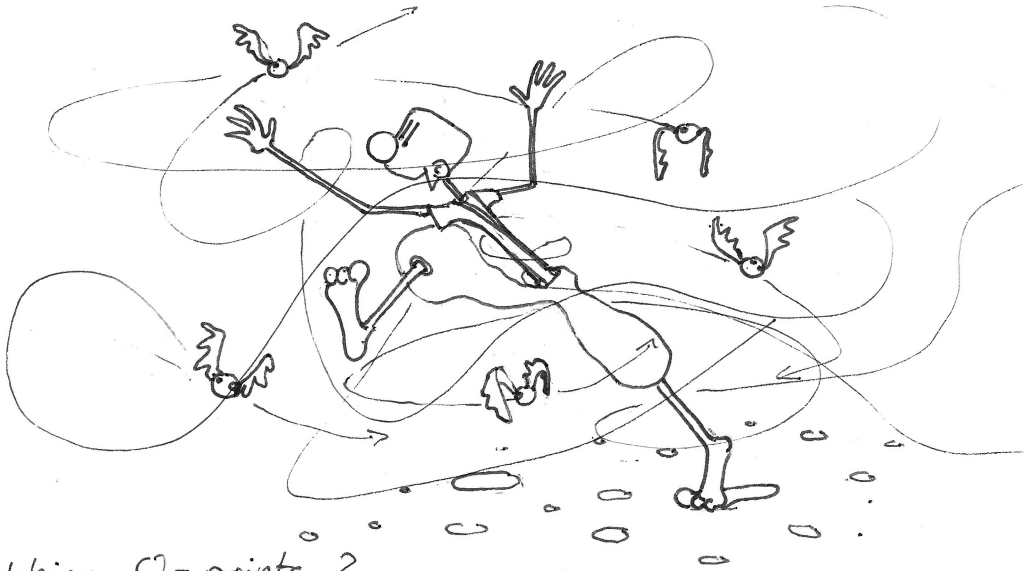
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Catching Q-points ?

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REPORT ON MY RESEARCH ON SOME REGULARITY QUESTIONS REGARDING MULTIVALUED/ Q -VALUED FUNCTIONS

JONAS HIRSCH

INTRODUCTION

In his pioneering work [1], F. Almgren developed a theory of multivalued functions. He introduced them as Q -valued functions. $Q \in \mathbb{N}$, fixed, indicates the number of values the function takes, counting multiplicity. We will refer to them from now on as Q -valued functions. Their purpose had been the development of a proof of a regularity result on area minimizing rectifiable currents.

F. Almgren considered it to be the "most pressing" problem in geometric measure theory and he considered the theory of Q -valued functions as an essential tool as the following quote indicates:

"Its solution has required development of several new geometric and analytic techniques, central among which is utilization of Q -valued functions to study branching phenomena."¹

The necessity of understanding branching and the essential motivation introducing multivalued functions can be demonstrated by the following example in complex function theory:

Consider the complex variety $\mathcal{V} = \{(z, v) \in \mathbb{C}^2 : z^2 = v^3\}$. Observe that $(0, 0)$ is a true branch point in the sense that there does not exist an open neighbourhood U of $(0, 0)$ s.t. $\mathcal{V} \cap U$ is an immersed surface. $\mathcal{V} \cap U$ is always a multivalued graph. To every $z = re^{i\theta}$ one has the three roots $v_j = r^{\frac{2}{3}} e^{i(\frac{2\theta}{3} + j\frac{2\pi}{3})}$, $j = 0, 1, 2$. One cannot define a root function that is continuous on the whole complex domain \mathbb{C} , because assuming such a continuous function exists and following the path $t \in [0, 1] \mapsto e^{i2\pi t}$ around zero will end up on an other sheet contradicting the continuity. But the complex variety \mathcal{V} can be considered as a two dimension surface in $\mathbb{R}^4 \cong \mathbb{C}^2$. Based on a classical computation of Wirtinger [24], Federer observed that complex varieties are calibrated and therefore area minimizers in their homology class [6, 16, section 5.4.19]. This example demonstrates that branching occurs for area minimizers in higher codimensions. If one allows a function to take several values, something like " $z \mapsto \{r^{\frac{2}{3}} e^{i(\frac{2\theta}{3} + j\frac{2\pi}{3})}\}_{j=0,1,2}$ ", \mathcal{V} can still be written as a graph over a flat disc.

Following a groundbreaking idea of De Giorgi in the single valued setting, F. Almgren uses that the first order term in the Taylor expansion of the area integrand for Q -valued Lipschitz-graph corresponds with the Dirichlet energy. Therefore he considers in his work [1] mainly Dirichlet minimizing Q -valued functions. Nonetheless he was thinking already about other possible applications as the following quote shows:

¹J. Almgren, [1], page 1

"It is one of the objects of the study of multivalued functions to provide a setting in which very general surfaces can be represented as images of simple domains (such as disks) under multivalued mappings. One of the main reasons for wanting to do this is to be able to utilize functional analytic techniques in novel ways in the geometric study of such surfaces;..."²

My research was mainly concerned with the study of general regularity questions of Q -valued functions. This is a report about the outcome, three independent results. Therefore a report structure is chosen and each can be read and understood on its own. To facilitate this each part starts with its own introduction. There we present its results, structure and put into context to other works.

The common thread are Q -valued functions. Since there are nonetheless some further links between them we give now a brief overview. For example in every part a regularity questions in this context is addressed. More precisely my research was focused on two main aspects: boundary regularity for Dirichlet minimizers and interior regularity for Q -valued harmonic maps.

Part 0

We give an overview on Q -valued functions. We restrict us to the essentials needed to understand the three other parts. So this part recalls the basic definitions and results omitting the proofs. More refined definitions and results are presented at the places where they are actually needed. This is done to keep this part as short and easy as possible. For a more detailed and a more or less complete picture we recommend [12].

Part 1

We consider the Hölder continuity for the Dirichlet problem at the boundary. Almgren introduced the Q -valued functions for studying regularity of minimal surfaces in higher codimension. The Hölder continuity in the interior for Dirichlet minimizers is an outcome of Almgren's original theory [1], to which C. De Lellis and E.N. Spadaro's work have given a simpler alternative approach [12]. This part extends the Hölder regularity for Dirichlet minimizing Q -valued functions up to the boundary assuming C^1 regularity of the domain and $C^{0,\alpha}$ regularity of the boundary data with $\alpha > \frac{1}{2}$.

Part 2

We present examples of holomorphic functions that vanish to infinite order at points at the boundary of their domain of definition. They give rise to examples of Dirichlet minimizing Q -valued functions indicating that "higher"-regularity boundary results are difficult. Furthermore we discuss some implication to branching and vanishing phenomena in the context of minimal surfaces, Q -valued functions and unique continuation.

Part 3

We consider multivalued maps into a smooth, compact Riemannian manifold locally minimizing the Dirichlet energy. An interior partial Hölder regularity result in the spirit of R. Schoen and K. Uhlenbeck is presented. Consequently a minimizer is Hölder continuous outside a set of Hausdorff dimension at most $N - 3$. As mentioned before, F. Almgren's original theory includes a global interior Hölder continuity result if the minimizers are valued into some \mathbb{R}^m . It cannot hold in

²J. Almgren, [1], page 8

general if the target is changed into a Riemannian manifold, since it already fails for "classical" single valued harmonic maps.

Part 0. Short overview of some general results on Q -valued functions

0.1. INTRODUCTION

This preamble recalls the basic definitions and results on Q -valued functions needed to understand each of the three parts. The theory is presented omitting the actual proofs. They can be found for instance in C. De Lellis and E. Spadaro's work [12]. More refined results and definitions will be introduced at places where these are actually needed. For example the appendix to part 1 contains an interpolation lemma in the spirit of Luckhaus with boundary functions in a fractional Sobolev space and a $W^{s,p}$, $s > \frac{1}{2}$ selection criterion and the appendix to part 3 contains an intrinsic proof the "classical" Luckhaus lemma. In the appendix to this preamble we present a concentration compactness result. It is along the same lines and indeed inspired by C. De Lellis and E. Spadaro's version [14, Lemma 3.2].

0.2. Q -VALUED FUNCTIONS

We follow mainly the notation and terminology introduced by C. De Lellis and E. Spadaro in [12]. It differs slightly from Almgren's original one. Q, Q_1, Q_2, \dots are always natural numbers.

The space of unordered sets of Q points in \mathbb{R}^n can be made into a complete metric space.

Definition 0.2.1. $(\mathcal{A}_Q(\mathbb{R}^n), \mathcal{G})$ denotes the metric space of unordered Q -tuples given by

$$\mathcal{A}_Q(\mathbb{R}^n) = \left\{ T = \sum_{i=1}^Q \llbracket t_i \rrbracket : t_i \in \mathbb{R}^n, i = 1, \dots, Q \right\}$$

and if \mathcal{P}_Q is the permutation group of $\{1, \dots, Q\}$ the metric is given by

$$\mathcal{G}(S, T)^2 = \min_{\sigma \in \mathcal{P}_Q} \sum_{i=1}^Q |s_i - t_{\sigma(i)}|^2.$$

We use the convention $\llbracket t \rrbracket = \delta_t$ for a Dirac measure at a point $t \in \mathbb{R}^n$. Considering $T = \sum_{i=1}^Q \llbracket t_i \rrbracket$ as a sum of Q Dirac measures one notice that $\mathcal{A}_Q(\mathbb{R}^n)$ corresponds to the set of 0-dimensional integral currents of mass Q and positive orientation. Hence we will write

$$\text{spt}(T) = \{t_1, \dots, t_Q : T = \sum_{i=1}^Q \llbracket t_i \rrbracket\} \subset \mathbb{R}^n.$$

Furthermore $\mathcal{A}_Q(\mathbb{R}^n)$ is endowed with an intrinsic addition:

$$+ : \mathcal{A}_{Q_1}(\mathbb{R}^n) \times \mathcal{A}_{Q_2}(\mathbb{R}^n) \rightarrow \mathcal{A}_{Q_1+Q_2}(\mathbb{R}^n) \quad S + T = \sum_{i=1}^{Q_1} \llbracket s_i \rrbracket + \sum_{i=1}^{Q_2} \llbracket t_i \rrbracket.$$

We define a translation operator

$$\oplus : \mathcal{A}_Q(\mathbb{R}^n) \times \mathbb{R}^n \rightarrow \mathcal{A}_Q(\mathbb{R}^n) \quad T \oplus s = \sum_{i=1}^Q \llbracket t_i + s \rrbracket.$$

The metric \mathcal{G} defines continuity, modulus of continuity, Hölder and Lipschitz continuity and (Lebesgue) measurability for functions from a set $\Omega \subset \mathbb{R}^N$ into $\mathcal{A}_Q(\mathbb{R}^n)$, i.e. $u : \Omega \rightarrow \mathcal{A}_Q(\mathbb{R}^n)$.

As it has been shown in [12, Proposition 0.4] for any measurable function $u : \Omega \rightarrow \mathcal{A}_Q(\mathbb{R}^n)$ we can find a measurable selection i.e.

$$v = (v_1, \dots, v_Q) : \Omega \rightarrow (\mathbb{R}^n)^Q \text{ measurable s.t. } u(x) = [v](x) = \sum_{i=1}^Q \llbracket v_i(x) \rrbracket.$$

Selections of higher regularity are considered in [11], [12, Proposition 1.2] and in the appendix B.3.

We will write $|u(x)| = \sqrt{\sum_{i=1}^Q |v_i(x)|^2} = \mathcal{G}(u(x), Q\llbracket 0 \rrbracket)$.

Definition 0.2.2. The Sobolev space $W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ is defined as the set of measurable functions $u : \Omega \rightarrow \mathcal{A}_Q(\mathbb{R}^n)$ that satisfy

- (w1) $x \mapsto \mathcal{G}(u(x), T) \in W^{1,2}(\Omega, \mathbb{R}_+)$ for every $T \in \mathcal{A}_Q(\mathbb{R}^n)$;
- (w2) $\exists \varphi_j \in L^2(\Omega, \mathbb{R}_+)$ for $j = 1, \dots, N$ s.t. $|D_j \mathcal{G}(u(x), T)| \leq \varphi_j(x)$ for any $T \in \mathcal{A}_Q(\mathbb{R}^n)$ and a.e. $x \in \Omega$.

It is not difficult to show the existence of minimal functions $\tilde{\varphi}_j$, in the sense that $\tilde{\varphi}_j(x) \leq \varphi_j(x)$ for a.e. x and any φ_j satisfying property (w2), [12, Proposition 4.2]. Such a minimal bound is denoted by $|D_j u|$ and is explicitly characterised by

$$|D_j u|(x) = \sup \{ |D_j \mathcal{G}(u(x), T_i)| : \{T_i\}_{i \in \mathbb{N}} \text{ dense in } \mathcal{A}_Q(\mathbb{R}^n) \}.$$

The Sobolev "semi-norm", or Dirichlet energy, is defined by integrating the measurable function $|Du|^2 = \sum_{j=1}^N |D_j u|^2$:

$$(0.2.1) \quad \int_{\Omega} |Du|^2 = \int_{\Omega} \sum_{j=1}^J |D_j u|^2.$$

Strictly speaking it is not a "semi-norm". $W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ is not a linear space since $\mathcal{A}_Q(\mathbb{R}^n)$ lacks this property.

A function $u \in W^{1,2}(\Omega, \mathbb{R}^n)$ is said to be Dirichlet minimizing if

$$(0.2.2) \quad \int_{\Omega} |Du|^2 = \inf \left\{ \int_{\Omega} |Dv|^2 : v \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n)), \mathcal{G}(u(x), v(x)) \in W_0^{1,2}(\Omega, \mathbb{R}_+) \right\}.$$

On Lipschitz regular domains $\Omega \subset \mathbb{R}^N$ one has a continuous trace operator as for classical single valued Sobolev functions

$$\circ|_{\partial\Omega} : W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n)) \rightarrow L^2(\partial\Omega, \mathcal{A}_Q(\mathbb{R}^n)).$$

The definition of $W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$, definition 0.2.2, implies that on a Lipschitz regular domain $\Omega \subset \mathbb{R}^N$ one has that $\mathcal{G}(u(x), v(x)) \in W_0^{1,2}(\Omega)$ corresponds to $u|_{\partial\Omega} = v|_{\partial\Omega}$ for any $u, v \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$.

As a consequence of a Rademacher theorem for multivalued Lipschitz functions, [12, section 1.3 & Theorem 1.13] a Sobolev function $u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ is a.e. approximately differentiable in the sense

- (1) $\exists \mathcal{U}_x : \Omega \rightarrow \mathcal{A}_Q(\mathbb{R}^n \times \text{Hom}(\mathbb{R}^N, \mathbb{R}^n))$, $x \mapsto \mathcal{U}_x = \sum_{i=1}^Q \llbracket (u_i(x), U_i(x)) \rrbracket$ measurable with $U_i(x) = U_j(x)$ whenever $u_i(x) = u_j(x)$;
- (2) \mathcal{U}_x defines a 1-jet $J\mathcal{U}_x : \Omega \times \mathbb{R}^N \rightarrow \mathcal{A}_Q(\mathbb{R}^n)$ by $J\mathcal{U}_x(y) = \sum_{i=1}^Q \llbracket u_i(x) + U_i(x)(y - x) \rrbracket$, that has the additional property that $J\mathcal{U}_x(x) = u(x)$ for a.e. $x \in \Omega$;
- (3) for a.e. $x \in \Omega$, $\exists E_x \subset \Omega$ having density 1 in x s.t. $\mathcal{G}(u(y), J\mathcal{U}_x(y)) = o(|y - x|)$ on E_x .

As one may guess the 1-jet corresponds to a first order "Taylor expansion", that becomes apparent in the proof of Rademacher's theorem, [12, Theorem 1.13]. One can show that $|D_j u|(x) = \sum_{i=1}^Q |U_i(x) e_j|^2$ for a.e. $x \in \Omega$, [12, Proposition 2.17]. From now on we will write $Du_i(x)$ for $U_i(x)$ and $D_j u_i(x)$ for $U_i(x) e_j$. For a definition of $C^k(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ see section 2.7.2.

A useful tool is Almgren's bi-Lipschitz embedding of $\mathcal{A}_Q(\mathbb{R}^n)$ into some \mathbb{R}^N . A remark of Brian White improved it, compare [12, Theorem 2.1 & Corollary 2.2]:

Theorem 0.2.1 (bi-Lipschitz embedding). *There exists $m = m(Q, n)$ and an injective map $\xi : \mathcal{A}_Q(\mathbb{R}^n) \rightarrow \mathbb{R}^m$ with the properties*

- (i) $Lip(\xi) \leq 1$ and $Lip(\xi^{-1}|_{\xi(\mathcal{A}_Q(\mathbb{R}^n))}) \leq C(Q, n)$;
- (ii) $\forall T \in \mathcal{A}_Q(\mathbb{R}^n) \exists \delta = \delta(T) > 0$ such that $|\xi(T) - \xi(S)| = \mathcal{G}(T, S)$ for all $S \in B_\delta(T) \subset \mathcal{A}_Q(\mathbb{R}^n)$.

There is a retraction $\rho : \mathbb{R}^m \rightarrow \mathcal{A}_Q(\mathbb{R}^n)$ because of (i) and the Lipschitz extension Theorem, e.g. [12, Theorem 1.7].

As a consequence $|Du|(x) = |D\xi \circ u|(x)$ for a.e. $x \in \Omega$ for any $u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$. We want to remark that the image of $\mathcal{A}_Q(\mathbb{R}^n)$ under ξ in \mathbb{R}^m is not convex neither a C^2 manifold. Thus there is no "nearest point" projection not even in a tubular neighborhood.

Two cornerstones in the context of Dirichlet minimizers that are of interest for us in the following are (c.p. with [12, Theorem 0.8 & Theorem 0.9]): .

Theorem 0.2.2 (Existence of Dirichlet minimizers). *Let $v \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ be given, then there exists a (not necessarily unique) Dirichlet minimizing $u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ with $\mathcal{G}(u(x), v(x)) \in W_0^{1,2}(\Omega, \mathbb{R}_+)$.*

Theorem 0.2.3 (interior Hölder continuity). *There is a constant $\alpha_0 = \alpha_0(N, Q) > 0$ with the property that if $u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ is Dirichlet minimizing, then $u \in C^{0,\alpha_0}(K, \mathcal{A}_Q(\mathbb{R}^n))$ for any $K \subset \Omega \subset \mathbb{R}^N$ compact. Indeed, $|Du|$ is an element of the Morrey space $L^{2,N-2-2\alpha_0}$ with the estimate*

$$(0.2.3) \quad r^{2-N-2\alpha_0} \int_{B_r(x)} |Du|^2 \leq R^{2-N-2\alpha_0} \int_{B_R(x)} |Du|^2 \text{ for } r \leq R, B_R(x) \subset \Omega.$$

For two-dimensional domains $\alpha_0(2, Q) = \frac{1}{Q}$ is explicit and optimal.

Both results had been proven first by Almgren in [1] and nicely reviewed by C. De Lellis and E. Spadaro in [12].

J. Almgren presents in [1, Theorem 2.16] an example of non-uniqueness: there are two Dirichlet minimizers $f \neq h \in W^{1,2}(B_1, \mathcal{A}_2(\mathbb{R}^2))$, $B_1 \subset \mathbb{R}^2$, with $f = h$ on ∂B_1 . Given any other minimizer that agrees with f or h at the boundary must be either f or h .

APPENDIX A. CONCENTRATION COMPACTNESS FOR Q -VALUED FUNCTIONS

Let $\Omega \subset \mathbb{R}^N$ be given, then there is a concentration compactness lemma for sequences $u(k) \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ with uniformly bounded energy.

Lemma A.1. *Given a sequence $u(k) \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ and a sequence of means $T(k) \in \mathcal{A}_Q(\mathbb{R}^n)$ with*

$$\limsup_{k \rightarrow \infty} \int_{\Omega} |Du(k)|^2 \leq \infty \text{ and } \int_{\Omega} \mathcal{G}(u(k), T(k))^2 \leq C \int_{\Omega} |Du(k)|^2$$

for a subsequence, not relabelled, we can find:

- (i) maps $b_l \in W^{1,2}(\Omega, \mathcal{A}_{Q_l}(\mathbb{R}^n))$ for $l = 1, \dots, J$, $\sum_{l=1}^L Q_l = Q$;
- (ii) a splitting $T(k) = T_1(k) + \dots + T_L(k)$ with $T_l(k) \in \mathcal{A}_{Q_l}(\mathbb{R}^n)$ and
 - $\limsup_k \text{diam}(\text{spt}(T_l(k))) < \infty$ for all $l = 1, \dots, L$
 - $\lim_{k \rightarrow \infty} \text{dist}(\text{spt}(T_l(k)), \text{spt}(T_m(k))) = \infty$ for $l \neq m$;
- (iii) a sequence $t_l(k) \in \text{spt}(T_l(k))$ such that $\mathcal{G}(u(k), b(k)) \rightarrow 0$ in L^2 with $b(k) = \sum_{l=1}^L (b_l \oplus t_l(k))$.

Moreover, the following two additional properties hold:

- (a) if $\Omega' \subset \Omega$ is open and A_k is a sequence of measurable sets with $|A_k| \rightarrow 0$, then

$$\liminf_{k \rightarrow \infty} \int_{\Omega' \setminus A_k} |Du(k)|^2 - \int_{\Omega'} |Db(k)|^2 \geq 0.$$

- (b) $\liminf_{k \rightarrow \infty} \int_{\Omega} (|Du(k)|^2 - |Db(k)|^2) = 0$ if and only if $\liminf_{k \rightarrow \infty} \int_{\Omega} (|Du(k)| - |Db(k)|)^2 = 0$.

Before we give the proof we recall the definition of the separation $\text{sep}(T)$ of a Q -point $T = \sum_{i=1}^Q \llbracket t_i \rrbracket \in \mathcal{A}_Q(\mathbb{R}^n)$.

$$\text{sep}(T) = \begin{cases} 0, & \text{if } T = Q \llbracket t \rrbracket \\ \min_{t_i \neq t_j} |t_i - t_j|, & \text{otherwise.} \end{cases}$$

The following results are of essential use in the context of the separation and needed for the proof of the concentration compactness lemma. The first gives a kind of relation between $\text{diam}(\text{spt}(T))$ and $\text{sep}(T)$, see [12, lemma 3.8]; the second gives a retraction $\boldsymbol{\vartheta} = \boldsymbol{\vartheta}_T$ based on $\text{sep}(T)$, see [12, lemma 3.7]

Lemma A.2. *To every $\epsilon > 0$ there exists $\beta = \beta(\epsilon, Q) > 0$ with the property that to any $T \in \mathcal{A}_Q(\mathbb{R}^n)$ there exists $S = S(T) \in \mathcal{A}_Q(\mathbb{R}^n)$ with*

$$\text{spt}(S) \subset \text{spt}(T), \quad \mathcal{G}(T, S) < \epsilon \text{sep}(S) \text{ and } \beta \text{diam}(\text{spt}(T)) < \text{sep}(S).$$

(For example $\beta = \epsilon^Q 3^{4-Q^2}$ works.)

Lemma A.3. *To a given $T \in \mathcal{A}_Q(\mathbb{R}^n)$ and $0 < 4s < \text{sep}(T)$ there exists a 1-Lipschitz retraction*

$$\boldsymbol{\vartheta} = \boldsymbol{\vartheta}_T : \mathcal{A}_Q(\mathbb{R}^n) \rightarrow \overline{B_s(T)} = \{S \in \mathcal{A}_Q(T) : \mathcal{G}(S, T) \leq s\}$$

with the property that

- (i) $\boldsymbol{\vartheta}(S) = S$ if $\mathcal{G}(S, T) \leq s$;
- (ii) $\mathcal{G}(\boldsymbol{\vartheta}(S_1), \boldsymbol{\vartheta}(S_2)) < \mathcal{G}(S_1, S_2)$ if $\mathcal{G}(S_1, T) > s$.

Proof of lemma A.1. We distinguish two cases. The second will be handled by induction on the first.

Case 1 and basis of the induction: $\liminf_{k \rightarrow \infty} \text{diam}(\text{spt}(T(k))) < \infty$
($\text{diam}(\text{spt}(T(k))) = 0$ for $Q = 1$):

Passing to an appropriate subsequence, not relabelled $\text{diam}(\text{spt}(T(k))) < C$ for all k . Set $L = 1$, and as splitting keep the sequence itself i.e. $T(k) = T_1(k)$. To every k fix a $t_1(k) \in \text{spt}(T(k))$.

Hence we have

$$\begin{aligned} \limsup_k \int_{\Omega} |u(k) \oplus (-t_1(k))|^2 &= \limsup_k \int_{\Omega} \mathcal{G}(u(k), Q \llbracket t_1(k) \rrbracket)^2 \\ &\leq \limsup_k 2 \int_{\Omega} \mathcal{G}(u(k), T(k))^2 + 2|\Omega| \mathcal{G}(T(k), Q \llbracket t_1(k) \rrbracket)^2 < \infty. \end{aligned}$$

Hence passing to an appropriate subsequence there is $b = b_1 \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ with $u(k) \oplus (-t_1(k)) \rightarrow b$ in L^2 . This proves (i),(ii),(iii), since $\mathcal{G}(u(k) \oplus -t_1(k), b) = \mathcal{G}(u(k), b \oplus t_1(k)) = \mathcal{G}(u(k), b(k))$. Furthermore, the established properties imply that $\xi \circ u(k) \rightarrow \xi \circ b(k)$ in $W^{1,2}(\Omega, \mathbb{R}^m)$. The additional property (a) follows, because $\mathbf{1}_{\Omega' \setminus A_k} \rightarrow \mathbf{1}_{\Omega'}$ in $L^2(\Omega)$ and so $\mathbf{1}_{\Omega' \setminus A_k} D\xi \circ u(k) \rightarrow \mathbf{1}_{\Omega'} D\xi \circ b(k)$. Property (b) holds because $L^2(\Omega)$ is an Hilbert space. Therefore we have, that $f_k = D\xi \circ u(k) \rightarrow f = D\xi \circ b(k)$ in $L^2(\Omega)$ if and only if $f_k \rightharpoonup f$ and $\|f_k\|_{L^2(\Omega)}^2 \rightarrow \|f\|_{L^2(\Omega)}^2$; compare $\liminf_k \|f_k - f\|^2 = \liminf_k \|f_k\|^2 + \|f\|^2 - 2\langle f_k, f \rangle = \liminf_k \|f_k\|^2 - \|f\|^2$.

Case 2 and the induction step: $\liminf_k \text{diam}(\text{spt}(T(k))) = +\infty$
Suppose the lemma holds for $Q' < Q$. To every $T(k)$ pick $S(k) \in \mathcal{A}_Q(\mathbb{R}^n)$ using A.2 s.t. for $S(k) = \sum_{j=1}^{J(k)} Q_j(k) \llbracket s_j(k) \rrbracket \in \mathcal{A}_Q(\mathbb{R}^n)$ set $\sigma_k = \text{sep}(S(k))$, then $\beta(\frac{1}{10}, Q) \text{diam}(\text{spt}(T(k))) < \sigma_k$ and $\mathcal{G}(T(k), S(k)) < \frac{\sigma_k}{10}$. Passing to an appropriate subsequence, not relabelled, we may further assume that $J(k) > 1$ and $Q_j(k)$ do not depend on k . Fix the associated 1-Lipschitz retractions of A.3 $\boldsymbol{\vartheta}_k : \mathcal{A}_Q(\mathbb{R}^n) \rightarrow \overline{B_{\frac{1}{5}S(k)}(S(k))}$ i.e. $\mathcal{H}^0(\text{spt}(\boldsymbol{\vartheta}_k(T)) \cap B_{\frac{\sigma_k}{5}}(s_j)) = Q_j$ for all $T \in \mathcal{A}_Q(\mathbb{R}^n)$ and $j = 1, \dots, J$. Hence these retractions $\boldsymbol{\vartheta}_k$ defines new sequences $v_j(k)$ in $W^{1,2}(\Omega, \mathcal{A}_{Q_j}(\mathbb{R}^n))$ and a splitting of $T(k)$:

$$\begin{aligned} \boldsymbol{\vartheta}_k \circ u(k) &= v_1(k) + \dots + v_J(k) \text{ with } v_j(k) \in B_{\frac{\sigma_k}{5}}(s_j); \\ T(k) &= \boldsymbol{\vartheta}_k \circ T(k) = T_1(k) + \dots + T_J(k) \text{ with } T_j(k) \in B_{\frac{\sigma_k}{5}}(s_j) \end{aligned}$$

Each sequence $v_j(k)$, $j = 1, \dots, J$ satisfies itself the assumptions of the lemma, because $\boldsymbol{\vartheta}_k$ is a retraction and so

$$(A.1) \quad \sum_{j=1}^J |Dv_j(k)|^2 = |D\boldsymbol{\vartheta}_k \circ u(k)|^2 \leq |Du(k)|^2$$

$$(A.2) \quad \sum_{j=1}^J \mathcal{G}(v_j(k), T_j(k))^2 = \mathcal{G}(\boldsymbol{\vartheta}_k \circ u(k), \boldsymbol{\vartheta}_k \circ T(k))^2 \leq \mathcal{G}(u(k), T(k))^2.$$

Furthermore we record some properties:

Defining $A_k = \{x : \boldsymbol{\vartheta}_k \circ u(k)(x) \neq u(k)(x)\} = \{x : \mathcal{G}(u(k), S(k)) > \frac{\sigma_k}{5}\} \subset \{x : \mathcal{G}(u(k), T(k)) \geq \frac{\sigma_k}{10}\} = B_k$ (subsets of Ω) we have

(1.) $|B_k| \rightarrow 0$ as $k \rightarrow \infty$, because

$$\begin{aligned} |B_k| &\leq \left(\frac{10}{\sigma_k}\right)^{2^*} \int_{B_k} \mathcal{G}(u(k), T(k))^{2^*} \\ &\leq \left(\frac{10}{\sigma_k}\right)^{2^*} C \left(\int_{\Omega} |Du(k)|^2\right)^{\frac{2^*}{2}} \rightarrow 0; \end{aligned}$$

(2.) $\mathcal{G}(u(k), \boldsymbol{\vartheta}_k \circ u(k)) \rightarrow 0$ in L^2 as $k \rightarrow \infty$, since

$$\begin{aligned} \int_{\Omega} \mathcal{G}(u(k), \boldsymbol{\vartheta}_k \circ u(k))^2 &= \int_{A_k} \mathcal{G}(u(k), \boldsymbol{\vartheta}_k \circ u(k))^2 \\ &\leq 2 \int_{B_k} \mathcal{G}(v_k, T(k))^2 + \mathcal{G}(\boldsymbol{\vartheta}_k \circ u(k), \boldsymbol{\vartheta}_k \circ T(k))^2 \\ &\leq 4 \left(\frac{10}{\sigma_k}\right)^{2^* - 2} \int_{B_k} \mathcal{G}(u(k), T(k))^{2^*} \\ &\leq \frac{C}{\sigma_k^{2^* - 2}} \left(\int_{\Omega} |Du(k)|^2\right)^{\frac{2^*}{2}} \rightarrow 0; \end{aligned}$$

- (3.) $\text{dist}(spt(T_i), spt(T_j)) \geq \sigma_k - 2\mathcal{G}(S(k), T(k)) \geq \frac{4}{5}\sigma_k \rightarrow +\infty$ for any $i \neq j$ as $k \rightarrow \infty$;
- (4.) $\|Du(k) - |D\boldsymbol{\vartheta}_k \circ u(k)|\| \rightarrow 0$ in L^2 as $k \rightarrow \infty$, because $|B_k| \rightarrow 0$, $|D\boldsymbol{\vartheta}_k \circ u(k)| \leq |Du(k)|$, $D\boldsymbol{\vartheta}_k \circ u(k) = Du(k)$ on $\Omega \setminus B_k$ and

$$\begin{aligned} \int_{\Omega} (|Du(k)| - |D\boldsymbol{\vartheta}_k \circ u(k)|)^2 &\leq \int_{\Omega} |Du(k)|^2 - |D\boldsymbol{\vartheta}_k \circ u(k)|^2 \\ &= \int_{B_k} |Du(k)|^2 - |D\boldsymbol{\vartheta}_k \circ u(k)|^2 \leq \int_{B_k} |Du(k)|^2 \rightarrow 0. \end{aligned}$$

Due to the induction hypothesis the lemma holds for each sequence $v_j(k)$ i.e. we can find $b_{j,l} \in W^{1,2}(\Omega, \mathcal{A}_{Q_{j,l}}(\mathbb{R}^n))$, with $\sum_{l=1}^{L_j} Q_{j,l} = Q_j$, a splitting $T_j(k) = T_{j,1}(k) + \dots + T_{j,L_j}(k)$ together with sequences $t_{j,l}(k) \in spt(T_{j,l}(k))$ satisfying the conditions (i), (ii), (iii). Furthermore the additional properties (a), (b) hold. Set $L = \sum_{j=1}^J L_j$, $K_j = \sum_{i=1}^{j-1} L_i$ and relabel $b_{K_j+l} = b_{j,l}$, $T_{K_j+l}(k) = T_{j,l}(k)$, $t_{K_j+l}(k) = t_{j,l}(k)$ and $Q_{K_j+l} = Q_{j,l}$ for $j \in \{1, \dots, J\}$ and $l \in \{1, \dots, L_j\}$. The induction hypothesis on the lemma states that the obtained sequences b_l , $T_l(k)$, $t_l(k)$ for $l = 1, \dots, L$ satisfy

- (i) $b_l \in W^{1,2}(\Omega, \mathcal{A}_{Q_l}(\mathbb{R}^n))$ for $l = 1, \dots, L$ and $\sum_{l=1}^L Q_l = Q$;
- (ii) $T(k) = T_1(k) + \dots + T_L(k)$, $t_l(k) \in spt(T_l(k))$ and
 $-\limsup_k \text{diam}(spt(T_l(k))) < \infty$ for all $l = 1, \dots, L$
 $-\lim_{k \rightarrow \infty} \text{dist}(spt(T_l(k)), spt(T_m(k))) = \infty$ for $l \neq m$ for any $K_j < l < m \leq K_{j+1}$, $j = 1, \dots, J$
- (iii) $\mathcal{G}(v_j(k), b_j(k)) \rightarrow 0$ in L^2 with $b_j(k) = \sum_{l=K_j+1}^{K_{j+1}} (b_l \oplus t_l(k))$ for each j .

Moreover, the following two additional properties hold for each j :

- (a) if $\Omega' \subset \Omega$ is open and A_k is a sequence of measurable sets with $|A_k| \rightarrow 0$, then

$$\liminf_{k \rightarrow \infty} \int_{\Omega' \setminus A_k} |Dv_j(k)|^2 - \int_{\Omega'} |Db_j(k)| \geq 0.$$

- (b) $\liminf_{k \rightarrow \infty} \int_{\Omega} (|Dv_j(k)|^2 - |Db_j(k)|^2) = 0$ if and only if
 $\liminf_{k \rightarrow \infty} \int_{\Omega} (|Dv_j(k)| - |Db_j(k)|)^2 = 0$.

Due to properties (1) to (4) we may sum in j and replace $\sum_{j=1}^J v_j(k)$ by $u(k)$. This completes the proof. \square

Part 1. Boundary regularity of Dirichlet minimizing Q -valued functions

1.2. INTRODUCTION

We address the following regularity question concerning Almgrens multivalued functions, posed for example by C.De Lellis in [15, section 8, (7)]:
Are Dirichlet minimizers continuous, or even Hölder, up to the boundary if the boundary data are sufficient regular?

The following result gives a rather general first answer:

Theorem 1.2.1. *Let $\frac{1}{2} < s \leq 1$ be given. There is a constant $\alpha = \alpha(N, Q, n, s) > 0$ with the property that, if*

- (a1) $\Omega \subset \mathbb{R}^N$ is a bounded C^1 regular domain;
- (a2) $u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ is Dirichlet minimizing;
- (a3) $u|_{\partial\Omega} \in C^{0,s}(\partial\Omega)$;

then $u \in C^{0,\alpha}(\overline{\Omega})$.

In terms of notation, for single valued functions, Sobolev spaces are denoted by $W^{1,p}(\Omega, \mathbb{R}^n)$ and $W^{1,p}(\Omega)$, fractional Sobolev spaces by $W^{s,p}(\Omega)$. In the case of multivalued functions we will always mention the target explicitly i.e. $W^{1,p}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ for Sobolev spaces and the fractional ones by $W^{s,p}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$. In the case of single valued function we will sometimes use as well $H^1(\Omega)$, $H^s(\Omega)$ for $W^{1,2}(\Omega)$ and $W^{s,2}(\Omega)$ ($p = 2$). The trace for a Sobolev function is denoted by $u|_{\partial\Omega}$. It will be clear from the context if it is the trace of a single valued or multivalued function.

The equivalent "classical" statement of Theorem 1.2.1 for single valued harmonic functions states:

$f : \Omega \rightarrow \mathbb{R}^n$ harmonic, $f|_{\partial\Omega} \in C^{0,\beta}(\partial\Omega)$ for some $0 < \beta < 1$ then $f \in C^{0,\beta}(\overline{\Omega})$.

Harmonic functions with finite energy belong to $H^1(\Omega, \mathbb{R}^n)$, but $u \in H^1(\Omega)$ if and only if $u|_{\partial\Omega} \in H^{\frac{1}{2}}(\partial\Omega)$. $H^{\frac{1}{2}}(\partial\Omega)$ can be characterised using the Gagliardo semi-norm $\int_{\partial\Omega \times \partial\Omega} \frac{|f(x) - f(y)|^2}{|x - y|^N} dx dy$ that is controlled by the $C^{0,\beta}(\partial\Omega)$ -norm for $\beta > \frac{1}{2}$. Nonetheless our result is suboptimal in the sense that for classical harmonic functions $u|_{\partial\Omega} \in W^{\frac{1}{2},2}(\partial\Omega) \cap C^{0,\beta}(\partial\Omega)$ for any $0 < \beta < 1$ implies $u \in C^{0,\beta}(\overline{\Omega})$. In contrast, the Hölder exponent we claim in Theorem 1.2.1 is not explicit. For dimension three and higher that is not really surprising since the optimal (or even an explicit) exponent is not known in the interior so far.

The result for two dimensions is somewhat unsatisfactory. In two dimensions the optimal Hölder exponent for the interior regularity for Q -valued Dirichlet minimizers is known and explicit: it is $\frac{1}{Q}$. We obtain the two dimensional case of theorem 1.2.1 by "lifting it" to three dimensions. So we get a "bad", not explicit exponent. Therefore we try to give some additional information. That continuity extends up to the boundary for two dimensional balls had been proven by W. Zhu in [25, Theorem 1.3]. We will give a proof on different lines that continuity extends up to the boundary of Lipschitz regular domains. Concerning the optimal exponent we can give a partial first answer. At least on conical subsets of Ω the interior regularity extends up to the boundary for boundary data $u|_{\partial\Omega} \in C^{0,\beta}(\partial\Omega)$, $\beta > \frac{1}{2}$.

The appendix contains a short introduction to fractional Sobolev spaces for single valued functions. It includes some perhaps less known results. Furthermore an interpolation lemma in the spirit of Luckhaus with boundaries functions in a fractional Sobolev space $W^{s,2}$ with $s > \frac{1}{2}$ is presented. Afterwards these results are

extended to Q -valued functions. Additionally we present a $W^{s,p}$ selection criterion, needed in the two dimensional setting.

Outline of this part: section 1.3 fixes notation and general assumptions, section 1.4 contains the proof of theorem 1.2.1 for dimension three and higher, section 1.5 considers the two dimensional setting. Finally the appendix with sections A, B and C provides tools needed in the proof.

1.3. GENERAL ASSUMPTIONS AND NOTATION

From now on, if not indicated differently, we will consider the following setting: $\Omega \subset \mathbb{R}^N$ is a bounded C^1 -regular domain i.e. to every $z \in \partial\Omega$ there exists $R = R(z) > 0$, $F = F_z \in C^1(\mathbb{R}^{N-1}, \mathbb{R})$ s.t. (up to a rotation)

$$\Omega \cap B_R(z) = \{z + (x', x_N) : |x| < R, x_N > F(x')\}.$$

In particular for $F \in C^1(\mathbb{R}^{N-1}, \mathbb{R})$ we set

$$\Omega_F = \{(x', x_N) : x_N > F(x')\}.$$

Since $\partial\Omega$ is compact, the C^1 regularity implies that

(A1) for any given $\epsilon_F > 0$, $\exists R = R(\Omega, \epsilon_F) > 0$ with the property that for any $z \in \Omega$ there is $F \in C^1(\mathbb{R}^{N-1}, \mathbb{R})$ with $F(0) = 0$, $\text{grad } F(0) = 0$, $\|\text{grad } F\|_\infty < \epsilon_F$ and (up to a rotation):

$$\Omega \cap B_R(z) = \{z + (x', x_N) : |x| < R, x_N > F(x')\} = \Omega_F \cap B_R.$$

In other words $\partial\Omega$ is locally the graph of a C^1 function with small gradient over the tangent space $T_z\partial\Omega$.

Let $0 < r \leq R$ and $z \in \partial\Omega$. We define the following scaled (and translated) Ω :

$$\Omega_{z,r} = \{x \in \mathbb{R}^N : z + rx \in \Omega\}.$$

Boundary regularity is a local question so we will often consider

$$\Omega_{z,r} \cap B_1 = \{(x', x_N) : |x| < 1, x_N > F_{0,r}(x')\} = \Omega_{F_{0,r}} \cap B_1$$

with $F_{0,r}(x') = r^{-1}F(rx')$ (observe that $\|\text{grad}(F_{0,r})\|_{\infty, B_1} = \|\text{grad } F\|_{\infty, B_r}$).

Frequently we will study such a special domain Ω_F defined by

(A2)

$$\Omega_F = \{(x', x_N) : x_N > F(x')\}$$

with $F \in C^1(\mathbb{R}^{N-1}, \mathbb{R})$ with $F(0) = 0$, $\text{grad } F(0) = 0$, $\|\text{grad } F\|_\infty < \epsilon_F$.

Moreover we set

$$\Gamma_F = \partial\Omega_F \cap B_1 = \{(x', x_N) : |x| < 1, x_N = F(x')\}.$$

Γ_F denotes a boundary portion of the boundary to such a special domain.

The upper half space \mathbb{R}_+^N is a particular case of such a domain i.e. $\Omega_0 = \mathbb{R}_+^N$ for $F = 0$. The boundary of the upper half ball $B_{1+} = \mathbb{R}_+^N \cap B_1$ is the union of $\Gamma_0 = B_1 \cap \{x_N = 0\}$ and the upper half of the sphere $\mathcal{S}_+^{N-1} = \mathcal{S}^{N-1} \cap \{x_N > 0\}$.

Fractional Sobolev spaces, named $W^{s,2}$, occur naturally, when dealing with boundary regularity for elliptic problems. A short introduction is given in the appendix A. We define the Gagliardo semi-norms for $0 < s < 1$ and m dimensional submanifolds $\Sigma \subset \mathbb{R}^N$

$$\begin{aligned} \|f\|_{s,\Sigma}^2 &= \int_{\Sigma \times \Sigma} \frac{|f(x) - f(y)|^2}{|x - y|^{m+2s}} dx dy, \quad f \in L^2(\Sigma) \\ \|u\|_{s,\Sigma}^2 &= \int_{\Sigma \times \Sigma} \frac{\mathcal{G}(u(x), u(y))^2}{|x - y|^{m+2s}} dx dy, \quad u \in L^2(\Sigma, \mathcal{A}_Q(\mathbb{R}^n)). \end{aligned}$$

The notation $\|\cdot\|_{s,\Sigma}$ has been chosen in similarity to the classical notation $[\cdot]_{\alpha,\Sigma}$ for the Hölder semi-norm with exponent α . We extend it to $s = 1$ by (abusing the notation a little):

$$\begin{aligned}\|f\|_{1,\Sigma}^2 &= \int_{\Sigma} |D_{\tau} f|^2, \quad f \in W^{1,2}(\Sigma) \\ \|u\|_{1,\Sigma}^2 &= \int_{\Sigma} |D_{\tau} u|^2, \quad u \in W^{1,2}(\Sigma, \mathcal{A}_Q(\mathbb{R}^n))\end{aligned}$$

where D_{τ} denotes the total tangential derivative on Σ . For single valued functions $f \in W^{1,2}(\Sigma)$ and an orthonormal frame τ_1, \dots, τ_m of $T_x \Sigma$ we have $|D_{\tau} f(x)|^2 = \sum_{j=1}^Q |\frac{\partial f}{\partial \tau_j}|^2$. In the case of multivalued function u we make use of the approximate differentiability of Sobolev functions: for a.e. $x \in \Sigma$ we have $|D_{\tau} u|^2(x) = \sum_{j=1}^m \sum_{i=1}^Q |U_i(x) \tau_j|^2$ where $U_i(x)$ are the elements of the 1-jet $\mathcal{J}u_x$, c.f. the discussion below definition 0.2.2 for precise statement to the approximate differentiability and the definition of the 1-jet.

1.4. HÖLDER CONTINUITY FOR $N \geq 3$

A more precise version of theorem 1.2.1 is:

Theorem 1.4.1. *For any $\frac{1}{2} < s \leq 1$, there are constants $C > 0$ and $\alpha_1 > 0$ depending on $N, n, Q, s, N \geq 3$ with the property that, if*

- (a1) $u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ is Dirichlet minimizing;
- (a2) $u|_{\partial\Omega} \in W^{s,2}(\partial\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ and for some $0 < \beta$ there is a constant $M_u > 0$ s.t.

$$r^{2(s-\beta)-(N-1)} \|u\|_{s, B_r(z) \cap \partial\Omega}^2 \leq M_u^2 \text{ for all } z \in \partial\Omega, r > 0;$$

then the following holds

- (i) $|Du|$ is an element of the Morrey space $L^{2, N-2+2\alpha}$ for any $0 < \alpha < \min\{\alpha_1, \beta\}$, more precisely the following estimate holds

$$(1.4.1) \quad r^{2-N-2\alpha} \int_{B_r(x) \cap \Omega} |Du|^2 \leq 2^N R_0^{2-N-2\alpha} \int_{B_{2R_0}(x) \cap \Omega} |Du|^2 + C \frac{R_0^{2(\beta-\alpha)}}{\beta-\alpha} M_u^2$$

for any $r < \frac{R_0}{2}$. The positive constant R_0 depends only on N, n, Q, s, Ω but not on the specific u ;

- (ii) $u \in C^{0,\alpha}(\overline{\Omega})$.

Lemma 1.4.2. *There is a relation between assumption (a2) and the Hölder continuity of $u|_{\partial\Omega}$:*

- (i) (a2) is satisfied if $u|_{\partial\Omega} \in C^{0,\beta}(\partial\Omega)$ for $\beta > \frac{1}{2}$ i.e. there is a dimensional constant $C > 0$ s.t. for $0 < s < \beta$

$$r^{2(s-\beta)-(N-1)} \|u\|_{s, B_r(z) \cap \partial\Omega}^2 \leq \frac{C}{\beta-s} [u]_{\beta, \partial\Omega}^2 \quad \forall z \in \partial\Omega, 0 < r < R(\Omega, 1);$$

- (ii) if (a2) holds then $u|_{\partial\Omega} \in C^{0,\beta}(\partial\Omega)$ i.e. there is a dimensional constant $C > 0$ s.t.

$$\mathcal{G}(u(x), u(y)) \leq CM|x-y|^\beta \quad \forall x, y \in \partial\Omega, |x-y| \leq \frac{R(\Omega, 1)}{2}.$$

Proof. To prove (i) let $z \in \partial\Omega$, $0 < r < R(\Omega, 1)$ be given and $F \in C^1(\mathbb{R}^{N-1}, \mathbb{R})$ the function of (A1), then

$$\begin{aligned}
& \int_{B_r(z) \cap \partial\Omega \times B_r(z) \cap \partial\Omega} \frac{\mathcal{G}(u(x), u(y))^2}{|x - y|^{N-1+2s}} dx dy \\
& \leq [u]_{\beta, \partial\Omega}^2 \int_{B_r(z) \cap \partial\Omega \times B_r(z) \cap \partial\Omega} |x - y|^{2(\beta-s)-(N-1)} dx dy \\
& \leq [u]_{\beta, \partial\Omega}^2 (1 + \|\text{grad}(F)\|_\infty^2) \int_{B_r \times B_r} |x' - y'|^{2(\beta-s)-(N-1)} dx' dy' \\
& \leq \frac{4(N-1)\omega_{N-1}^2}{2(\beta-s)} [u]_{\beta, \partial\Omega}^2 r^{2(\beta-s)+(N-1)}.
\end{aligned}$$

To prove (ii) we observe that using the function F of (A1) to write $\partial\Omega$ locally as a graph we can transform it to a local question on \mathbb{R}^{N-1} . Furthermore making use of Almgren's bilipschitz embedding, Theorem 0.2.1, it is sufficient to check it for single valued functions. Hence (ii) is equivalent to check that

There is a dimensional constant $C > 0$ s.t. if $f \in W^{s,2}(\mathbb{R}^N, \mathbb{R}^n)$ and $M_f > 0$ be given with the property that

$$(1.4.2) \quad r^{2(s-\beta)-N} \|f\|_{s, B_r(z)}^2 \leq M_f^2 \quad \forall B_r(z) \subset \mathbb{R}^N, 0 < r < R_0$$

then $f \in C^{0,\beta}(\mathbb{R}^N, \mathbb{R}^n)$ with

$$(1.4.3) \quad |f(x) - f(y)| \leq CM_f |x - y|^\beta \quad \forall |x - y| < R_0.$$

Let us write $f(z, r) = f_{B_r(z)} f$ for any $B_r(z) \subset \mathbb{R}^N$, then using twice Cauchy's inequality we have

$$\begin{aligned}
& \int_{B_r(z)} |f - f(z, r)| \leq |B_r(z)|^{-2} \int_{B_r(z) \times B_r(z)} |f(x) - f(y)| dx dy \\
& \leq |B_r(z)|^{-2} \int_{B_r(z)} \left(\int_{B_r(z)} |x - y|^{N+2s} dy \right)^{\frac{1}{2}} \left(\int_{B_r(z)} \frac{|f(x) - f(y)|^2}{|x - y|^{N+2s}} dy \right)^{\frac{1}{2}} dx \\
& \leq \left(\frac{4^N}{\omega_N^2} r^{2s-N} \|f\|_{s, B_r(z)}^2 \right)^{\frac{1}{2}} \leq Cr^\beta M_f.
\end{aligned}$$

Hence for any $r < R_0$ and $k \in \mathbb{N}$

$$|f(z, 2^{-k-1}r) - f(z, 2^{-k}r)| \leq 2^N \int_{B_{2^{-k}r}(z)} |f - f(z, 2^{-k}r)| \leq CM_f r^\beta 2^{-\beta k};$$

i.e. $k \mapsto f(z, 2^{-k}r)$ is a Cauchy sequence because $\sum_{k=0}^\infty |f(z, 2^{-k-1}r) - f(z, 2^{-k}r)| \leq \frac{CM_f}{1-2^{-\beta}} r^\beta$. Furthermore for any $z_1, z_2 \in \mathbb{R}^N$ with $|z_1 - z_2| = r < R_0$ we find

$$\begin{aligned}
|f(z_1) - f(z_2)| & \leq \sum_{i=1}^2 |f(z_i) - f(z_i, r)| + \int_{B_r(z_i) \cap B_r(z_2)} |f(x) - f(z_i)| dx \\
& \leq \sum_{i=1}^2 \frac{CM_f}{1-2^{-\beta}} r^\beta + \frac{CM_f}{1-2^{-\beta}} r^\beta \leq 4 \frac{CM_f}{1-2^{-\beta}} r^\beta;
\end{aligned}$$

this shows that $f \in C^{0,\beta}$. □

The core of the proof of theorem 1.4.1 is the estimate stated in proposition 1.4.3 below. To make its proof more accessible it is presented in the next subsection and split into several lemmas.

Proposition 1.4.3. *For any $\frac{1}{2} < s \leq 1$ there are constants $\epsilon_0 > 0$, $0 < \delta < \frac{1}{N-2}$ and $C > 0$ depending on N, n, Q, s with the property that, if (A2) holds with $\epsilon_F \leq \epsilon_0$, then*

$$(1.4.4) \quad \int_{\Omega_F \cap B_1} |Du|^2 \leq \left(\frac{1}{N-2} - \delta \right) \int_{S^{N-1} \cap \Omega_F} |D_\tau u|^2 + C \|u\|_{s, \Gamma_F}^2.$$

for any Dirichlet minimizer $u \in W^{1,2}(B_1 \cap \Omega_F, \mathcal{A}_Q(\mathbb{R}^n))$.

Let us take the previous proposition, i.e. the estimate (1.4.4), for granted and close the argument in the proof of theorem 1.4.1.

Proof of Theorem 1.4.1. Let ϵ_0, δ be the constants of proposition 1.4.3. Fix $\alpha_1 \leq \alpha_0$ (α_0 being the Hölder exponent of theorem 0.2.2) s.t. $(N-2+2\alpha_1) \left(\frac{1}{N-2} - \delta \right) \leq 1$. Let $R_0 = R_0(\Omega, \epsilon_0)$ be the radius defined of (A1) for $\epsilon_F = \epsilon_0$

Due to the choice of R_0 , for any $0 < r \leq R_0$, $z \in \partial\Omega$ the rescaled map

$$u_{z,r}(x) = u(z + rx) \quad \text{for } x \in B_1 \cap \Omega_{z,r}$$

belongs to $W^{1,2}(\Omega_{z,r} \cap B_1, \mathcal{A}_Q(\mathbb{R}^n))$ and satisfies the assumptions of the proposition 1.4.3. One readily checks that for $\frac{1}{2} < s \leq 1$

$$\|u_{z,r}\|_{s, B_1 \cap \partial\Omega_{z,r}}^2 = r^{2s-(N-1)} \|u\|_{s, B_r(z) \cap \partial\Omega}^2.$$

Applying (1.4.4) and assumption (a2) we get

$$\begin{aligned} r^{2-N} \int_{B_r(z) \cap \Omega} |Du|^2 &= \int_{B_1 \cap \Omega_{z,r}} |Du_{z,r}|^2 \\ &\leq \left(\frac{1}{N-2} - \delta \right) \int_{S^{N-1} \cap \Omega_{z,r}} |D_\tau u_{z,r}|^2 + C \|u_{z,r}\|_{s, B_1 \cap \partial\Omega_{z,r}}^2 \\ &\leq \frac{1}{N-2+2\alpha_1} r^{3-N} \int_{\partial B_r(z) \cap \Omega} |D_\tau u|^2 + C r^{2\beta} M_u^2. \end{aligned}$$

Hence for a.e. $0 < r < R_0$ and $0 < \alpha < \min\{\alpha_1, \beta\}$

$$\begin{aligned} & - \frac{\partial}{\partial r} \left(r^{2-N-2\alpha} \int_{B_r(z) \cap \Omega} |Du|^2 \right) \\ &= -r^{2-N-2\alpha} \int_{\partial B_r(z) \cap \Omega} |Du|^2 + (N-2+2\alpha) r^{-1-2\alpha} r^{2-N} \int_{B_r(z) \cap \Omega} |Du|^2 \\ &\leq r^{2-N-2\alpha} \int_{\partial B_r(z) \cap \Omega} |D_\tau u|^2 - |Du|^2 + (N-2+2\alpha) C r^{2(\beta-\alpha)-1} M_u^2 \\ &\leq (N-2+2\alpha) C r^{2(\beta-\alpha)-1} M_u^2. \end{aligned}$$

Integrating in r we achieve the following inequality for any $z \in \partial\Omega$ and $0 < r \leq R_0$:

$$(1.4.5) \quad r^{2-N-2\alpha} \int_{B_r(z) \cap \Omega} |Du|^2 - R_0^{2-N-2\alpha} \int_{B_{R_0}(z) \cap \Omega} |Du|^2 \leq \frac{C}{\beta-\alpha} R_0^{2(\beta-\alpha)} M_u^2.$$

Now we can conclude (1.4.1). If $x \in \bar{\Omega}$ satisfies $\text{dist}(x, \partial\Omega) > \frac{R_0}{2}$, then $B_r(x) \subset B_{\frac{R_0}{2}}(x) \subset \Omega$ for any $0 < r < \frac{R_0}{2}$ and so, by (0.2.3) in Theorem 0.2.3

$$\begin{aligned} (1.4.6) \quad r^{2-N-2\alpha} \int_{B_r(x)} |Du|^2 &\leq \left(\frac{R_0}{2} \right)^{2-N-2\alpha} \int_{B_{\frac{R_0}{2}}(x)} |Du|^2 \\ &\leq 2^N R_0^{2-N-2\alpha} \int_{B_{2R_0}(x) \cap \Omega} |Du|^2. \end{aligned}$$

Assume therefore $x \in \overline{\Omega}$ has $\text{dist}(x, \partial\Omega) \leq \frac{R_0}{2}$. Fix $z \in \partial\Omega$ s.t. $\text{dist}(x, \partial\Omega) = |x - z|$, and for $0 < r \leq \frac{R_0}{2}$ set $r_1 = \max\{r, |x - z|\}$, $r_2 = r_1 + |x - z| \leq 2r_1 \leq R_0$. Then

$$\begin{aligned}
 (1.4.7) \quad & r^{2-N-2\alpha} \int_{B_r(x) \cap \Omega} |Du|^2 \leq r_1^{2-N-2\alpha} \int_{B_{r_1}(x) \cap \Omega} |Du|^2 \\
 & \leq \left(\frac{r_2}{r_1}\right)^{N-2+2\alpha} r_2^{2-N-2\alpha} \int_{B_{r_2}(z) \cap \Omega} |Du|^2 \\
 & \leq 2^N \left(R_0^{2-N-2\alpha} \int_{B_{R_0}(z) \cap \Omega} |Du|^2 + \frac{C}{\beta - \alpha} R_0^{2(\beta - \alpha)} M_u^2 \right) \\
 & \leq 2^N \left(R_0^{2-N-2\alpha} \int_{B_{2R_0}(x) \cap \Omega} |Du|^2 + \frac{C}{\beta - \alpha} R_0^{2(\beta - \alpha)} M_u^2 \right).
 \end{aligned}$$

The fact (ii) i.e. $u \in C^{0,\alpha}(\overline{\Omega})$ follows now classically. We established that $|Du|$ is an element of the Morrey space $L^{2,N-2+2\alpha}(\Omega)$. Ω is C^1 regular and therefore by Poincaré's inequality this implies that $\xi \circ u$ is an element of the Campanato space $\mathcal{L}^{2,N+2\alpha}(\Omega)$, see for instance [10, Proposition 3.7]. Furthermore $\mathcal{L}^{2,N+2\alpha}(\Omega) = C^{0,\alpha}(\overline{\Omega})$, [10, Theorem 2.9]. \square

1.4.1. Proof of Proposition 1.4.3. The proof can be subdivided into two parts:
paragraph 1.4.1.1:

We show that it is necessary and sufficient for a Dirichlet minimizer on the upper half ball $B_1 \cap \{x_N > 0\}$ to be trivial that it has constant boundary data on $B_1 \cap \{x_N = 0\}$.

paragraph 1.4.1.2:

We show that if proposition would fail we could construct a non-trivial Dirichlet minimizer on the upper half ball $B_1 \cap \{x_N > 0\}$ with constant boundary data contradicting the previous step.

1.4.1.1. Non-existence of certain non-trivial minimizers. This paragraph is devoted to establish the following two results for certain Dirichlet minimizers on the upper half ball $B_{1+} = B_1 \cap \{x_N > 0\}$, recalling that $\mathcal{S}_+^{N-1} = \mathcal{S}^{N-1} \cap \{x_N > 0\}$ and $\Gamma_0 = B_1 \cap \{x_N = 0\}$.

Proposition 1.4.4. *Every 0-homogeneous Dirichlet minimizer in B_{1+} with $u|_{\Gamma_0} = \text{const.}$ is trivial i.e. constant.*

Corollary 1.4.5. *A Dirichlet minimizer on B_{1+} with $u|_{\Gamma_0} = \text{const.}$ satisfying*

$$(1.4.8) \quad \int_{B_{1+}} |Du|^2 = \frac{1}{N-2} \int_{\mathcal{S}_+^{N-1}} |D_\tau u|^2$$

needs to be constant.

They are both consequence of an appropriately chosen inner variation:

Lemma 1.4.6 (a special kind of inner variation). *Given a Dirichlet minimizer $u \in W^{1,2}(B_{1+}, \mathcal{A}_Q(\mathbb{R}^n))$ with $u|_{\Gamma_0} = \text{const.}$ and a vector field $X = (X_1, \dots, X_N) \in C_c^1(B_1, \mathbb{R}^N)$ with $e_N \cdot X(x', 0) = X_N(x', 0) \geq 0$ on Γ_0 , then*

$$(1.4.9) \quad 0 \leq \int_{B_{1+}} |Du|^2 \text{div}(X) - 2 \sum_{i=1}^Q \langle Du_i : Du_i DX \rangle.$$

Proof. Let u and X be given and set $T = u|_{\Gamma_0}(x)$ for $x \in \Gamma_0$. Observe that $x_N + tX_N(x', x_N) = x_N + t(X_N(x', x_N) - X_N(x', 0)) + tX_N(x', 0) \geq (1-t\|DX_N\|_\infty)x_N + tX_N(x', 0) \geq 0$ for $x_N > 0$ and sufficient small $0 < t < t_0$. Then for $t_0 > 0$ small

$$\Phi_t(x) = x + tX(x)$$

defines a 1-parameter family of C^1 -diffeomorphism that satisfy

$$A_t = \Phi_t(B_{1+}) \subset B_{1+} \text{ for } 0 \leq t \leq t_0.$$

So

$$v_t(x) = \begin{cases} u \circ \Phi_t^{-1}(x) & \text{for } x \in A_t \\ T & \text{for } x \in B_{1+}^+ \setminus A_t \end{cases}$$

defines a C^1 family of competitors to u . Standard calculations give

$$\begin{aligned} D\Phi_t^{-1} \circ \Phi_t &= (D\Phi_t)^{-1} = \sum_{k=0}^{\infty} (-t)^k (DX)^k = 1 - tDX + o(t) \\ \det(D\Phi_t) &= 1 + t \operatorname{div}(X) + o(t) \end{aligned}$$

so that

$$\begin{aligned} |Dv_t|^2 \circ \Phi_t &= \sum_{i=1}^Q |Du_i D\Phi_t^{-1} \circ \Phi_t|^2 = \sum_{i=1}^Q |Du_i (1 - tDX + o(t))|^2 \\ &= \sum_{i=1}^Q |Du_i|^2 - 2t \sum_{i=1}^Q \langle Du_i : Du_i DX \rangle + o(t). \end{aligned}$$

In total we found that for all $0 \leq t \leq t_0$

$$\begin{aligned} \int_{B_{1+}} |Dv_t|^2 &= \int_{A_t} |Dv_t|^2 = \int_{B_{1+}} |Dv_t|^2 \circ \Phi_t |\det D\Phi_t| \\ &= \int_{B_{1+}} |Du|^2 + t \int_{B_{1+}} |Du|^2 \operatorname{div}(X) - 2 \sum_{i=1}^Q \langle Du_i : Du_i DX \rangle + o(t). \end{aligned}$$

Since $\int_{B_{1+}} |Dv_t|^2 \geq \int_{B_{1+}} |Du|^2$, we necessarily have

$$0 \leq \int_{B_{1+}} |Du|^2 \operatorname{div}(X) - 2 \sum_{i=1}^Q \langle Du_i : Du_i DX \rangle.$$

□

Proof of Proposition 1.4.4. u being 0-homogeneous implies that $u(x) = u(\frac{x}{|x|})$ for a.e. x . Thus $\frac{\partial u}{\partial r}(x) = 0$ for a.e. $x \in B_{1+}$, which corresponds to

$$(1.4.10) \quad 0 = \frac{\partial u}{\partial r}(x) = \sum_{i=1}^Q \left[\sum_{j=1}^N D_j u_i(x) \frac{x_j}{|x|} \right].$$

Fix $0 < R < 1$ and consider the vector field $X(x) = \eta(|x|)e_N = (0, \dots, \eta(|x|))$ with

$$\eta(r) = \begin{cases} 1 - \frac{r}{R} & r \leq R \\ 0 & r \geq R. \end{cases}$$

Thus we have $X_N(x) \geq 0$ and $DX(x) = \eta'(|x|)e_N \otimes \frac{x}{|x|}$. This gives $\operatorname{div}(X)(x) = \eta'(|x|)\frac{x_N}{|x|}$ and due to (1.4.10)

$$\langle Du_i : Du_i DX \rangle = \sum_{j=1}^N \left\langle \frac{x_j}{|x|} D_j u_i, D_N u_i \right\rangle \eta'(|x|) = 0 \text{ for a.e. } x.$$

Using $\eta'(|x|) = -\frac{1}{R}\mathbf{1}_{B_R}(x)$ and applying Lemma 1.4.6 we get

$$0 \leq -\frac{1}{R} \int_{B_{R+}} |Du|^2 \frac{x_N}{|x|}.$$

This is only possible for $|Du| = 0$ on B_{R+} and so $|Du| = 0$ on B_{1+} . \square

Proof of corollary 1.4.5. Let $u \in W^{1,2}(B_{1+}, \mathcal{A}_Q(\mathbb{R}^n))$ be as assumed. Observe that (1.4.8) implies that $u \in W^{1,2}(\mathcal{S}_+^{N-1}, \mathcal{A}_Q(\mathbb{R}^n))$. Hence $v(x) = u\left(\frac{x}{|x|}\right)$ defines a 0-homogeneous competitor using $u|_{\Gamma_0} = \text{const.}$

$$\int_{B_{1+}} |Dv|^2 = \frac{1}{N-2} \int_{\mathcal{S}_+^{N-1}} |D_\tau v|^2 = \frac{1}{N-2} \int_{\mathcal{S}_+^{N-1}} |D_\tau u|^2 = \int_{B_{1+}} |Du|^2.$$

where we used firstly the 0-homogeneity of v , then $u|_{\mathcal{S}_+^{N-1}} = v|_{\mathcal{S}_+^{N-1}}$ and finally (1.4.8). Therefore v has to be minimizing as well, and moreover $Dv = 0$ as a consequence of proposition 1.4.4. This proves the corollary since then $Du = 0$ as well. \square

1.4.1.2. *contradiction argument.* In this section we want to establish by contradiction the estimate of Proposition 1.4.3

$$\int_{\Omega_F \cap B_1} |Du|^2 \leq \left(\frac{1}{N-2} - \delta \right) \int_{\mathcal{S}^{N-1} \cap \Omega_F} |D_\tau u|^2 + C \|u\|_{s, \Gamma_F}^2.$$

To prove Theorem 1.4.1 from such an estimate we only needed the scaling property $\|u_{z,r}\|_{s, B_1 \cap \partial\Omega_{z,r}}^2 = r^{2s-(N-1)} \|u\|_{s, B_r(z) \cap \partial\Omega}^2$ and the existence of positive constants $\beta, M_u > 0$ both depending possibly on u s.t. in combination $\|u_{z,r}\|_{s, B_1 \cap \partial\Omega_{z,r}} \leq r^\beta M_u$.

Before coming to the proof we discuss some subtleties in the strategy. A $C^{0,\beta}$ -Hölder norm, $[u]_{\beta, \Sigma} = \sup_{x,y \in \Sigma} \frac{\mathcal{G}(u(x), u(y))}{|x-y|^\beta}$, for any $0 < \beta < 1$ shares this property since

$$[u_{r,z}]_{\beta, \partial\Omega_{z,r} \cap B_1} = r^\beta [u]_{\beta, \partial\Omega \cap B_1(z)} \leq r^\beta [u]_{\beta, \partial\Omega}.$$

Replacing the $W^{s,2}(\partial\Omega)$ -norm, ($s > \frac{1}{2}$) by a Hölder-norm with exponent $\beta < \frac{1}{2}$ would be desirable since it would get us closer to the already mentioned classical result: $u \in W^{1,2}(\Omega)$ harmonic with $u|_{\partial\Omega} \in C^{0,\beta}(\partial\Omega)$ for some $\beta > 0$ implies $u \in C^{0,\beta}(\overline{\Omega})$.

Nonetheless we cannot hope to prove an estimate like (1.4.4) by contradiction if the fractional Sobolev norm ($s > \frac{1}{2}$) is replaced by an $C^{0,\beta}$ -Hölder norm, $\beta < \frac{1}{2}$ because vanishing of energy through the boundary needs to be excluded. Bounds on $W^{s,2}(\partial\Omega)$ -, or $C^{0,s}(\partial\Omega)$ -norms with $s < \frac{1}{2}$ are insufficient. This is demonstrated by the following two dimensional example on the disc $B_1 \subset \mathbb{R}^2$. It uses polar coordinates $x = \begin{pmatrix} r \cos(\theta) \\ r \sin(\theta) \end{pmatrix} = re^{i\theta}$.

Example 1.4.1. For any $\epsilon > 0$ there is a sequence of harmonic functions $f_k \in W^{1,2}(B_1, \mathbb{R})$ a positive constant $c > 0$ with the following properties: for all k we have $\int_{B_1} |Df_k|^2 > c$, $f_k(e^{i\theta}) = 0$ for $|\theta| > \epsilon$. Furthermore $f_k \rightarrow 0$ uniformly on B_1 and $\|f_k\|_{s, S^1}, [f_k]_{s, S^1} \rightarrow 0$ for every $s < \frac{1}{2}$.

Proof of example 1.4.1. To a given $0 < \epsilon < \frac{\pi}{2}$, fix a smooth, symmetric, non-negative bump function η with $\eta(0) > 0$ and $\eta(\theta) = 0$ for $|\theta| \geq \epsilon$. Let $\sum_{l=0}^{\infty} a_l \cos(l\theta)$ be the Fourier series of $\eta(\theta)$. It is converging uniformly to η in the C^∞ topology since η is smooth and $\sum_{l=0}^{\infty} l^m |a_l| < \infty$ for all $m \in \mathbb{N}$. Fix $k_0 \in \mathbb{N}$ sufficient large s.t. $2|a_k| < a_0 = \eta(0)$ for $k \geq k_0$ and set $A = \sum_{l=0}^{\infty} (l+1)|a_l| \geq (\sum_{l=0}^{\infty} (l+1)a_l^2)^{\frac{1}{2}}$.

The addition theorem $2 \cos(l\theta) \cos(k\theta) = \cos((l+k)\theta) + \cos((l-k)\theta)$ shows that the harmonic extension of $2\eta(\theta) \cos(k\theta)$ in B_1 is

$$\begin{aligned} g_k(r e^{i\theta}) &= \sum_{l=0}^{\infty} a_l \left(r^{l+k} \cos((l+k)\theta) + r^{|l-k|} \cos((l-k)\theta) \right) \\ &= \sum_{m=0}^{\infty} (a_{m-k} + a_{m+k}) r^m \cos(m\theta) \quad \text{with } a_{m-k} = 0 \text{ for } m < k. \end{aligned}$$

For $k \geq k_0$

$$\begin{aligned} \frac{1}{\pi} \int_{B_1} |Dg_k|^2 &= \sum_{m=1}^{\infty} m(a_{m-k} + a_{m+k})^2 \\ &\geq k(a_0 + a_{2k})^2 \geq \frac{1}{4} k a_0^2 \\ &\leq 2 \sum_{l=0}^{\infty} (l+k) a_l^2 + |l-k| a_l^2 \leq 4kA^2. \end{aligned}$$

We consider now the sequence of harmonic functions on B_1 given by $f_k(x) = \frac{g_k(x)}{k^{\frac{1}{2}}} \in W^{1,2}(B_1)$. f_k has the desired properties: using the equivalence

- (i) $\frac{1}{4} a_0^2 \leq \frac{1}{\pi} \int_{B_1} |Df_k|^2 = \|f_k\|_{\frac{1}{2}, S^1}^2 \leq 4A^2$ for all $k \geq k_0$;
- (ii) $f_k(e^{i\theta}) = 0$ for $|\theta| > \epsilon$ and all k ;
- (iii) $\|f_k\|_{\infty} \leq \frac{2\|\eta\|_{\infty}}{k^{\frac{1}{2}}} \rightarrow 0$ as $k \rightarrow \infty$;
- (iv) for any $0 < s < \frac{1}{2}$

$$\begin{aligned} \|f_k\|_{s, S^1}^2 &= \sum_{m=0}^{\infty} \frac{m^{2s}}{k} (a_{m-k} + a_{m+k})^2 \leq 8k^{2s-1} A^2 \\ [f_k]_{s, S^1} &\leq \sum_{m=0}^{\infty} \frac{m^s}{k^{\frac{1}{2}}} |a_{m-k} + a_{m+k}| \leq 2k^{s-\frac{1}{2}} \sum_{l=0}^{\infty} (l+1) |a_l| \end{aligned}$$

converging to 0 as $k \rightarrow \infty$.

(iii) follows from the maximum principle on harmonic functions. The fact that the $W^{s,2}$ -norm on S^1 corresponds to the sum in (iii), i.e. the equivalence $H^s(S^1) = W^{s,2}(S^1)$, is the content of corollary A.13. It is straightforward to check that one has $[\varphi]_{\beta, S^1} \leq \sum_{l=0}^{\infty} l^{\beta} |c_l|$ for a converging Fourier series $\varphi(\theta) = \sum_{l=0}^{\infty} c_l \cos(l\theta)$. \square

Proof of proposition 1.4.3. If $u \notin W^{1,2}(\mathcal{S}^{N-1} \cap \Omega_F, \mathcal{A}_Q(\mathbb{R}^n)) \cap W^{s,2}(\Gamma_F, \mathcal{A}_Q(\mathbb{R}^n))$ the LHS of (1.4.4) is infinite and so there is nothing to prove. Hence assuming that the proposition would not hold, we can find sequences $F(k) \in C^1(\mathbb{R}^{N-1}, \mathbb{R})$ satisfying (A2) with $\epsilon_F < \frac{1}{k}$ and associated $u(k) \in W^{1,2}(B_1 \cap \Omega_{F(k)}, \mathcal{A}_Q(\mathbb{R}^n))$ failing (1.4.4) i.e.

$$(1.4.11) \quad \int_{\Omega_{F(k)} \cap B_1} |Du(k)|^2 > \left(\frac{1}{N-2} - \frac{1}{k} \right) \int_{\mathcal{S}^{N-1} \cap \Omega_{F(k)}} |D_{\tau} u(k)|^2 + k \|u(k)\|_{s, \Gamma_{F(k)}}^2.$$

We may assume that the LHS of (1.4.11) is 1 by dividing each $u(k)$ by its Dirichlet energy $\left(\int_{\Omega_{F(k)} \cap B_1} |Du(k)|^2 \right)^{-\frac{1}{2}}$. We also assume, w.l.o.g., $k > k_0 > 4$.

To every k we may fix a C^1 -diffeomorphism $G(k) : \overline{B_{1+}} \rightarrow \overline{\Omega_{F(k)} \cap B_1}$, arguing for example on the base of Lemma C.2. $F(k) \rightarrow F_0 = 0$ in C^1 as $k \rightarrow \infty$ and therefore $G(k), G(k)^{-1} \rightarrow \mathbf{1}$ in C^1 ($\mathbf{1}$ deontes the identity map on \mathbb{R}^N). We consider now instead of the sequence $u(k)$ itself the sequence $v(k) = u(k) \circ$

$G(k) \in W^{1,2}(B_{1+}, \mathcal{A}_Q(\mathbb{R}^n))$. $v(k)$ has up to order $o(1)$ the same properties as $u(k)$ since $G(k), G(k)^{-1} \rightarrow \mathbf{1}$ in C^1 i.e.

$$\begin{aligned}
 \int_{B_1^+} |Dv(k)|^2 &= (1 + o(1)) \int_{\Omega_{F(k)}} |Du(k)|^2 && \leq 1 + o(1); \\
 (1.4.12) \quad \int_{\mathcal{S}_+^{N-1}} |D_\tau v(k)|^2 &= (1 + o(1)) \int_{\mathcal{S}^{N-1} \cap \Omega_{F(k)}} |D_\tau u(k)|^2 && < \frac{1 + o(1)}{\frac{1}{N-2} - \frac{1}{k}} < 2N; \\
 \|v(k)\|_{s, \Gamma_0}^2 &= (1 + o(1)) \|u(k)\|_{s, \Gamma_{F(k)}}^2 && \leq \frac{1 + o(1)}{k} \leq \frac{1}{2k}.
 \end{aligned}$$

(1.4.11) with LHS=1 provides the upper bounds. The second and third show that $v(k)|_{\partial B_{1+}} \in W^{1,2}(\mathcal{S}_+^{N-1}, \mathcal{A}_Q(\mathbb{R}^n)) \cap W^{s,2}(\Gamma_0, \mathcal{A}_Q(\mathbb{R}^n))$.

To every k fix a mean $T(k) \in \mathcal{A}_Q(\mathbb{R}^n)$ and apply the concentration compactness Lemma A.1 to the sequences $v(k), T(k)$. For a subsequence $v(k')$ we can find maps $b_j \in W^{1,2}(B_{1+}, \mathcal{A}_{Q_j}(\mathbb{R}^n))$, sequences $t_j(k') \in \text{spt}(T(k'))$ and a splitting $T(k') = T_1(k') + \dots + T_J(k')$. We will prove now that the b_j satisfy also the following:

- (i) $b_j|_{\mathcal{S}_+^{N-1}} \in W^{1,2}(\mathcal{S}_+^{N-1}, \mathcal{A}_{Q_j}(\mathbb{R}^n))$ and $b_j|_{\Gamma_0} = \text{const.}$;
- (ii) $\int_{B_{1+}} |Db_j|^2 \leq \frac{1}{N-2} \int_{\mathcal{S}_+^{N-1}} |D_\tau b_j|^2$ for all j ;
- (iii) $b_j \in W^{1,2}(B_{1+}, \mathcal{A}_{Q_j}(\mathbb{R}^n))$ is Dirichlet minimizing and

$$\sum_{j=1}^J \int_{B_{1+}} |Db_j|^2 = \lim_{k' \rightarrow \infty} \int_{B_{1+}} |Dv(k')|^2 = \lim_{k' \rightarrow \infty} \int_{\Omega_{F_{k'}} \cap B_1} |Du(k')|^2 = 1.$$

From now on we use $b(k') = \sum_{j=1}^J (b_j \oplus t_j(k'))$ as in the proof of the concentration compactness result.

Proof of (i): The concentration compactness lemma states that $\xi \circ v(k') \rightharpoonup \xi \circ b(k')$ in $W^{1,2}(B_{1+}, \mathbb{R}^m)$ and $\xi \circ v(k') \rightarrow \xi \circ b(k')$ in $L^2(B_{1+}, \mathbb{R}^m)$. This implies that $\xi \circ v(k') \rightharpoonup \xi \circ b(k)$ in $W^{1,2}(\mathcal{S}_+^{N-1}, \mathbb{R}^m)$ and $\xi \circ v(k') \rightarrow \xi \circ b(k')$ in $L^2(\mathcal{S}_+^{N-1}, \mathbb{R}^m)$, because $\xi \circ v(k') \in W^{1,2}(\mathcal{S}_+^{N-1}, \mathbb{R}^m)$ is uniformly bounded as seen in (1.4.12). The lower semicontinuity of energy together with (1.4.12) then states

$$\begin{aligned}
 (1.4.13) \quad \frac{1}{N-2} \sum_{j=1}^J \int_{\mathcal{S}_+^{N-1}} |D_\tau b_j|^2 &= \frac{1}{N-2} \int_{\mathcal{S}_+^{N-1}} \sum_{j=1}^J |D_\tau \xi \circ b_j|^2 \\
 &\leq \liminf_{k' \rightarrow \infty} \left(\left(\frac{1}{N-2} - \frac{1}{k'} \right) \int_{\mathcal{S}_+^{N-1}} |D_\tau \xi \circ v(k')|^2 \right) \leq 1.
 \end{aligned}$$

$\mathcal{G}(v|_{\Gamma_0}(k'), b|_{\Gamma_0}(k')) \rightarrow 0$ in $L^2(\Gamma_0)$ due to the weak convergence in the interior. Hence due to dominated convergence for any $\delta > 0$ and (1.4.12)

$$\begin{aligned}
 &\sum_{j=1}^J \int_{\Gamma_0 \times \Gamma_0} \frac{\mathcal{G}(b_j|_{\Gamma_0}(x), b_j|_{\Gamma_0}(y))^2}{|x-y|^{N-1+2s}} \\
 &= \lim_{k' \rightarrow \infty} \int_{\Gamma_0 \times \Gamma_0} \frac{\mathcal{G}(v|_{\Gamma_0}(k')(x), v|_{\Gamma_0}(k')(y))^2}{|x-y|^{N-1+2s}} \leq \lim_{k' \rightarrow \infty} \frac{2}{k'} = 0;
 \end{aligned}$$

consequently $b_j|_{\Gamma_0} = \text{const.}$ for all j .

Proof of (ii): Having established (i), $a_j(x) = b_j\left(\frac{x}{|x|}\right) \in W^{1,2}(B_{1+}, \mathcal{A}_{Q_j}(\mathbb{R}^n))$ is well-defined and an admissible competitor.

$$\int_{B_{1+}} |Db_j|^2 \leq \int_{B_{1+}} |Da_j|^2 = \frac{1}{N-2} \int_{\mathcal{S}_+^{N-1}} |D_\tau a_j|^2 = \frac{1}{N-2} \int_{\mathcal{S}_+^{N-1}} |D_\tau b_j|^2$$

for every j due to the 0-homogeneity of a_j and $a_j|_{\mathcal{S}_+^{N-1}} = b_j|_{\mathcal{S}_+^{N-1}}$.

Proof of (iii): Let $G : \overline{B_1} \rightarrow \overline{B_{1+}}$ be the bilipschitz map constructed in Lemma C.1. $\|v(k') \circ G\|_{s, \mathcal{S}^{N-1}}$ is uniformly bounded: Firstly apply Corollary B.1 to estimate

$$\|v(k') \circ G\|_{s, \mathcal{S}^{N-1}} \leq C \left(\|v(k') \circ G\|_{s, \mathcal{S}^{N-1} \cap \{x_N > \frac{-1}{\sqrt{5}}\}} + \|v(k') \circ G\|_{s, \mathcal{S}^{N-1} \cap \{x_N < \frac{-1}{\sqrt{5}}\}} \right);$$

secondly G is bilipschitz and $G(\mathcal{S}^{N-1} \cap \{x_N > \frac{-1}{\sqrt{5}}\}) = \mathcal{S}_+^{N-1}$ and $G(\mathcal{S}^{N-1} \cap \{x_N < \frac{-1}{\sqrt{5}}\}) = \Gamma_0$, so that

$$\begin{aligned} \|v(k') \circ G\|_{s, \mathcal{S}^{N-1} \cap \{x_N > \frac{-1}{\sqrt{5}}\}} &\leq C \|v(k')\|_{s, \mathcal{S}_+^{N-1}} \\ \|v(k') \circ G\|_{s, \mathcal{S}^{N-1} \cap \{x_N < \frac{-1}{\sqrt{5}}\}} &\leq C \|v(k')\|_{s, \Gamma_0}; \end{aligned}$$

thirdly the interpolation property $\|f\|_{s, \mathcal{S}_+^{N-1}}^2 \leq C \int_{\mathcal{S}_+^{N-1}} |Df|^2$ gives

$$\|v(k')\|_{s, \mathcal{S}_+^{N-1}} \leq \|Dv(k')\|_{L^2(\mathcal{S}_+^{N-1})};$$

finally we combine all of them and use (1.4.12) to conclude

$$\|v(k') \circ G\|_{s, \mathcal{S}^{N-1}} \leq C \left(\|Dv(k')\|_{L^2(\mathcal{S}_+^{N-1})} + \|v(k')\|_{s, \Gamma_0} \right) \leq C(2N).$$

The same bound holds for $b(k') \circ G \in W^{s,2}(\mathcal{S}^{N-1}, \mathcal{A}_Q(\mathbb{R}^n))$ because of the lower semicontinuity of energy established in (1.4.13). Furthermore in the proof of (i) we showed that $\mathcal{G}(v(k'), b(k')) \rightarrow 0$ in $L^2(\mathcal{S}_+^{N-1})$ and $L^2(\Gamma_0)$, so that

$$\|\mathcal{G}(v(k') \circ G, b(k') \circ G)\|_{L^2(\mathcal{S}^{N-1})} = o(1).$$

Fix any small $\epsilon > 0$ and $R_\epsilon > 0$ determined by the interpolation Lemma B.2. So to every k' we can find $w(k') \in W^{s,2}(A_{1, R_\epsilon}, \mathcal{A}_Q(\mathbb{R}^n))$ on the annulus $A_{1, R_\epsilon} = B_1 \setminus B_{R_\epsilon}$ interpolating between $v(k') \circ G$ and $b(k') \circ G$. Hence $w(k')(x) = v(k') \circ G(x)$, $w(k')(R_\epsilon x) = b(k') \circ G(x)$ for all $x \in \mathcal{S}^{N-1}$ and

$$\begin{aligned} &\int_{A_{1, R_\epsilon}} |Dw(k')|^2 \\ &\leq \epsilon \left(\|v(k') \circ G\|_{s, \mathcal{S}^{N-1}}^2 + \|b(k') \circ G\|_{s, \mathcal{S}^{N-1}}^2 \right) + C \|\mathcal{G}(v(k') \circ G, b(k') \circ G)\|_{L^2(\mathcal{S}^{N-1})}^2 \\ &\leq \epsilon C 4N + Co(1). \end{aligned}$$

To check the minimizing property let $c_j \in W^{1,2}(B_{1+}, \mathcal{A}_{Q_j}(\mathbb{R}^n))$ be an arbitrary competitor to b_j for $j = 1, \dots, J$. Set $c(k') = \sum_{j=1}^J (c_j \oplus t_j(k'))$. For $0 < R \leq 1$ we denote the map $G \circ \frac{1}{R} \circ G^{-1}(x) = \frac{\epsilon_N}{2} + \frac{1}{R} \left(x - \frac{\epsilon_N}{2}\right)$ by ψ_R . So we found

$$\int_{C_R} |Dc(k') \circ \psi_R|^2 = R^{N-2} \int_{B_{1+}} |Dc(k')|^2 \leq \int_{B_{1+}} |Dc(k')|^2$$

with $C_R = \psi_R^{-1}(B_{1+}) \subset B_{1+}$. We define $C(k') \in W^{1,2}(B_{1+}, \mathcal{A}_Q(\mathbb{R}^n))$ considering $G(B_R) = C_R$ by

$$C(k') = \begin{cases} w(k') \circ G^{-1}, & \text{if } x \in B_{1+} \setminus C_{R_\epsilon} = G(A_{1, R_\epsilon}) \\ c(k') \circ \psi_{R_\epsilon}, & \text{if } x \in C_{R_\epsilon}. \end{cases}$$

$C(k') \circ G(k') \in W^{1,2}(\Omega_{F(k)} \cap B_1, \mathcal{A}_Q(\mathbb{R}^n))$ is now an admissible competitor to $u(k')$ and therefore

$$\begin{aligned} (1 - o(1)) \int_{B_{1+}} |Dv(k')|^2 &\leq \int_{\Omega_{F(k)} \cap B_1} |Du(k)|^2 \leq (1 + o(1)) \int_{B_{1+}} |DC(k')|^2 \\ &\leq (1 + o(1)) C \int_{A_{1,R_\epsilon}} |Dw(k')|^2 + (1 + o(1)) \int_{B_{1+}} |Dc(k')|^2 \\ &\leq C(\epsilon + Co(1)) + (1 + o(1)) \sum_{j=1}^J \int_{B_{1+}} |Dc_j|^2. \end{aligned}$$

Pass to the lim inf and apply the lower semicontinuity ensured by the concentration compactness Lemma A.1 to conclude

$$\sum_{j=1}^J \int_{B_{1+}} |Db_j|^2 \leq \liminf_{k' \rightarrow \infty} (1 - o(1)) \int_{B_{1+}} |Dv(k')|^2 \leq C\epsilon + \sum_{j=1}^J \int_{B_{1+}} |Dc_j|^2.$$

ϵ can be chosen arbitrary small and C is a dimensional constant so that b_j has to be Dirichlet minimizing for every $j = 1, \dots, J$. The strong convergence in energy follows choosing $c_j = b_j$ for every j in the inequality above.

The maps b_j constructed above with the properties (i),(ii),(iii) contradict corollary 1.4.5. Firstly we found due to (iii), that

$$\begin{aligned} \sum_{j=1}^J \int_{B_{1+}} |Db_j|^2 &= \lim_{k' \rightarrow \infty} \int_{\Omega_{F(k')} \cap B_1} |Du(k')|^2 \\ &\geq \lim_{k' \rightarrow \infty} \left(\frac{1}{N-2} - \frac{1}{k'} \right) \int_{\Omega_{F(k)} \cap \mathcal{S}^{N-1}} |D_\tau u(k')|^2 \\ &= \lim_{k' \rightarrow \infty} \left(\frac{1}{N-2} - \frac{1}{k'} \right) \int_{\mathcal{S}_+^{N-1}} |D_\tau v(k')|^2 \\ &\geq \frac{1}{N-2} \sum_{j=1}^J \int_{\mathcal{S}_+^{N-1}} |D_\tau b_j|^2. \end{aligned}$$

Combining this with (ii) gives, for $j = 1, \dots, J$

$$\int_{B_{1+}} |Db_j|^2 = \frac{1}{N-2} \int_{\mathcal{S}_+^{N-1}} |D_\tau b_j|^2.$$

Corollary 1.4.5 states now that $Db_j = 0$ on B_{1+} because $b_j|_{\Gamma_0} = \text{const.}$ by (i). This contradicts (iii), because $1 = \int_{\Omega_{F(k')} \cap B_1} |Du(k')|^2$ for all k' .

This contradiction proves that the proposition must hold. \square

1.5. BOUNDARY REGULARITY IN DIMENSION $N = 2$

1.5.1. Global Hölder regularity. In this section we will show that Theorem 1.4.1 extends directly to two dimensions. We can consider the two dimensional case as a special case of a certain minimizer on a three dimensional domain.

Lemma 1.5.1. *Let $u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ be a minimizer on a domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$, then $U(x, t) = u(x)$ is an element of $W^{1,2}(\Omega \times I, \mathcal{A}_Q(\mathbb{R}^n))$ for any bounded open interval $I \subset \mathbb{R}$. U is Dirichlet minimizing.*

Proof. Assuming the contrary there exists $V \in W^{1,2}(\Omega \times I, \mathcal{A}_Q(\mathbb{R}^n))$ with $V = U$ on the boundary of $\Omega \times I$ i.e. $(x, t) \mapsto \mathcal{G}(U(x, t), V(x, t)) \in W_0^{1,2}(\Omega \times I)$ and

$$(1.5.1) \quad \int_{\Omega \times I} |DV|^2 < \int_{\Omega \times I} |DU|^2 = |I| \int_{\Omega} |Du|^2;$$

the second equality actually shows that $U \in W^{1,2}(\Omega \times I, \mathcal{A}_Q(\mathbb{R}^n))$. Consider the subset $J \subset I$

$$J = \{t \in I : x \mapsto v_t(x) = V(x, t) \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n)) \text{ and } v_t|_{\partial\Omega} = u|_{\partial\Omega}\};$$

then by Fubini's theorem $|I \setminus J| = 0$.

Furthermore there must be a $t \in J$ with

$$(1.5.2) \quad \int_{\Omega} |Dv_t|^2 dx < \int_{\Omega} |Du|^2;$$

non existence would contradict (1.5.1) because then

$$|I| \int_{\Omega} |Du|^2 = \int_J \int_{\Omega} |Du|^2 dt \leq \int_J \int_{\Omega} |Dv_t|^2 dx dt = \int_{\Omega \times I} |DV|^2.$$

v_t for $t \in J$ satisfying (1.5.2) is an admissible competitor to u , but (1.5.2) violates the minimality of u . \square

Remark 1.5.1. The converse of this lemma holds as well in the following sense, if $u(x) \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ and $U(x, t) = u(x)$ is Dirichlet minimizing on $\Omega \times \mathbb{R}$ then u itself is minimizing in Ω , in the sense of compact perturbations:

$$\int_{\{U \neq V\}} |DU|^2 \leq \int_{\{U \neq V\}} |DV|^2$$

for all $V \in W^{1,2}(\Omega \times \mathbb{R}, \mathcal{A}_Q(\mathbb{R}^n))$ with $\{U \neq V\}$ compact.

This had been proven in [12], but for the sake of completeness we recall their proof in the appendix, Lemma B.3.

From now on Ω denotes a C^1 regular domain in \mathbb{R}^2 .

Theorem 1.5.2. *For any $\frac{1}{2} < s \leq 1$, there are constants $C > 0$ and $\alpha_1 > 0$ depending on n, Q, s with the property that,*

- (a1) $u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ Dirichlet minimizing;
- (a2) $u|_{\partial\Omega} \in W^{s,2}(\partial\Omega, \mathcal{A}_Q(\mathbb{R}^n));$

then the following holds

- (i) $|Du|$ is an element of the Morrey space $L^{2,2\alpha}$ for any $0 < \alpha < \min\{\alpha_1, s - \frac{1}{2}\}$, more precisely the following estimate holds

$$(1.5.3) \quad r^{-2\alpha} \int_{B_r(x) \cap \Omega} |Du|^2 \leq 2^7 R_0^{-2\alpha} \int_{B_{2R_0}(x) \cap \Omega} |Du|^2 + C \frac{R_0^{2s-1-2\alpha}}{2s-1-2\alpha} \|u\|_{\partial\Omega}^2$$

for any $r < \frac{R_0}{4}$. The positive R_0 depends only on n, Q, s, Ω but not on the specific u ;

- (ii) $u \in C^{0,\alpha}(\overline{\Omega})$.

Proof. Set $\Omega_I = \Omega \times]-2L, 2L[\subset \mathbb{R}^3$ for some large $L > 0$. The boundary portion $\partial\Omega \times]-L, L[$ is C^1 regular by assumption on the regularity of $\partial\Omega$. $U(x, t) = u(x)$ is

an element of $W^{1,2}(\Omega_I, \mathcal{A}_Q(\mathbb{R}^n))$ and Dirichlet minimizing as seen in lemma 1.5.1. For any $(z, t_0) \in \partial\Omega \times]-L, L[$ and $0 < r < L$ we found

$$\begin{aligned} r^{2(s-\beta)-2} \|U\|_{s, B_r(z, t_0) \cap \partial\Omega_I}^2 &\leq r^{2(s-\beta)-2} \|U\|_{s, (B_r(z) \cap \partial\Omega) \times]t_0-r, t_0+r[}^2 \\ &= r^{2(s-\beta)-2} \int_{B_r(z) \cap \partial\Omega \times B_r(z) \cap \partial\Omega} \int_{t_0-r}^{t_0+r} \frac{\mathcal{G}(u(x), u(y))^2}{(|x-y|^2 + (t_1-t_2)^2)^{\frac{2+2s}{2}}} dt_1 dt_2 dx dy \\ &\leq C 2r^{2(s-\beta)-1} \int_{B_r(z) \cap \partial\Omega \times B_r(z) \cap \partial\Omega} \frac{\mathcal{G}(u(x), u(y))^2}{|x-y|^{1+2s}} dx dy \leq 2C r^{2(s-\beta)-1} \|u\|_{s, \partial\Omega}^2. \end{aligned}$$

(We have applied above the following auxiliary calculation. Let $\alpha > 0$ and $J = [a, a + \delta]$. After the change of variables $t_1 = a + rx$, $t_2 = a + ry$, we have

$$\begin{aligned} \int_{J \times J} \frac{1}{(r^2 + (t_1 - t_2)^2)^{\frac{\alpha+1}{2}}} dt_1 dt_2 &= 2r^{1-\alpha} \int_{\substack{[0, \frac{\delta}{r}] \times [0, \frac{\delta}{r}] \\ x \geq y}} \frac{1}{(1 + (x - y)^2)^{\frac{\alpha+1}{2}}} dx dy \\ &= 2r^{1-\alpha} \int_0^{\frac{\delta}{r}} \int_0^{\frac{\delta}{r}-y} \frac{1}{(1 + z^2)^{\frac{\alpha+1}{2}}} dz dy \leq 2r^{-\alpha} \delta \int_0^\infty \frac{1}{(1 + z^2)^{\frac{\alpha+1}{2}}} \\ &= C |J| r^{-\alpha}. \end{aligned}$$

The dimensional constant $C = 2 \int_0^\infty \frac{1}{(1+z^2)^{\frac{\alpha+1}{2}}} \leq \frac{\alpha+1}{\alpha}$ is therefore finite.)

Combining all obtained estimates we found that U satisfies the assumption of theorem 1.4.1 with $\beta = s - \frac{1}{2}$ and $M_U = \|u\|_{s, \partial\Omega}$ in (a2).

Apply Theorem 1.4.1, in particular (1.4.1), to U on a point $(x, 0) \in \Omega \times]-L, L[$ with $r < \frac{R_0}{4} < L$. This gives the desired (1.5.3), because

$$\begin{aligned} r^{-2\alpha} \int_{B_r(x) \cap \Omega} |Du|^2 &= \frac{r^{-2\alpha}}{2r} \int_{-r}^r \int_{B_r(x) \cap \Omega} |DU|^2 \leq 2^2 (2r)^{-1-2\alpha} \int_{B_{2r}((x,0)) \cap \Omega_I} |DU|^2 \\ &\leq 2^5 \left(R_0^{-1-2\alpha} \int_{B_{2R_0}((x,0)) \cap \Omega_I} |DU|^2 + C \frac{R_0^{2(\beta-\alpha)}}{\beta-\alpha} M_U^2 \right) \\ &\leq 2^7 R_0^{-2\alpha} \int_{B_{2R_0}(x) \cap \Omega} |Du|^2 + C \frac{R_0^{2s-1-2\alpha}}{2s-1-2\alpha} \|u\|_{s, \partial\Omega}^2. \end{aligned}$$

(ii) i.e. $u \in C^{0,\alpha}(\bar{\Omega})$ now follows as outlined in the proof to theorem 1.4.1. \square

1.5.2. Continuity up to boundary. That continuity extends up to the boundary for 2-dimensional ball has been proven by W. Zhu in [25]. His idea is based on the Courant-Lebesgue lemma and can be modified to work on Lipschitz regular domains as well. We will give here a different proof, that on a first glimpse doesn't seem to be so restricted to the 2-dimensional setting as it is for Zhu's proof due to the Courant-Lebesgue lemma. Our proof uses an interplay of classical trace estimates and energy decay. We shortly recall the classical trace estimates and their proof. The proof here is taken from [23, Lemma 13.5]. As introduced in the general assumptions, section 1.3, we use the notation $\Omega_F = \{(x', x_N) : x_N > F(x')\}$ for $F : \mathbb{R}^{N-1} \rightarrow \mathbb{R}$.

Lemma 1.5.3. *For F Lipschitz continuous and $1 < p < \infty$, one has*

$$(1.5.4) \quad \left\| \frac{f(x', x_N) - f|_{\partial\Omega_F}(x')}{x_N - F(x')} \right\|_{L^p(\tilde{\Omega})} \leq \frac{p}{p-1} \left\| \frac{\partial f}{\partial x_N} \right\|_{L^p(\tilde{\Omega})} \quad \forall f \in W^{1,p}(\Omega_F, \mathbb{R});$$

and any subset $\tilde{\Omega} \subset \Omega_F$ of the following type:

$$\tilde{\Omega} = \{(x', x_N) : x' \in \Omega', F(x') < x_N < G(x')\}$$

$\tilde{\Omega} \subset \mathbb{R}^{N-1}$ and $G \geq F$ continuous.

Equivalently one has

$$(1.5.5) \quad \left\| \frac{\mathcal{G}(u(x'), x_N), u|_{\partial\Omega_F}(x'))}{x_N - F(x')} \right\|_{L^p(\tilde{\Omega})} \leq \frac{p}{p-1} \|D_N u\|_{L^p(\tilde{\Omega})} \quad \forall u \in W^{1,p}(\Omega_F, \mathcal{A}_Q(\mathbb{R}^n)).$$

Proof. For $p > 1$ Hardy's inequality, compare for instance with [23, Lemma 13.4], states that, if $h \in L^p(\mathbb{R}_+)$, $g(t) := \frac{1}{t} \int_0^t h(s) ds \in L^p(\mathbb{R}_+)$ satisfies

$$(1.5.6) \quad \|g\|_p \leq \frac{p}{p-1} \|f\|_p.$$

For $f \in C_c^1(\overline{\Omega_F})$ set

$$h(t) := \mathbf{1}_{[0, G(x') - F(x')]}(t) \frac{\partial f}{\partial x_N}(x', F(x') + t).$$

Apply Hardy's inequality to it and observe that for $0 < t < G(x') - F(x')$ and $t = x_N - F(x')$

$$g(t) = \frac{f(x', F(x') + t) - f(x', F(x'))}{t} = \frac{f(x', x_N) - f|_{\partial\Omega_F}(x')}{x_N - F(x')}.$$

Hence take the power p and integrate in $x' \in \Omega'$ to conclude (1.5.5). By a density argument the inequality extends to all of $W^{1,p}(\Omega_F)$.

For a Lipschitz continuous $u \in W^{1,p}(\Omega_F)$, we have $u|_{\partial\Omega_F}(x') = u(x', F(x'))$. $k(t) := \mathcal{G}(u(x'), F(x') + t)$ is Lipschitz continuous in t . Furthermore $k'(t) \leq |D_N u|(x', F(x') + t)$ for a.e. x' . Apply Hardy's inequality this time to $h(t) = \mathbf{1}_{[0, G(x') - F(x')]}(t) k'(t)$, take the power p and integrate in $x' \in \Omega'$. This shows (1.5.5) under the additional assumption that u is Lipschitz. It extends by density to all of $W^{1,p}(\Omega_F)$. \square

Proposition 1.5.4. *Given a Dirichlet minimizer $u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ on a Lipschitz regular domain $\Omega \subset \mathbb{R}^N$ that satisfies*

(a1) $u|_{\partial\Omega}$ is continuous;

(a2) $N = 2$ or

$$(1.5.7) \quad r^{2-N} \int_{B_r(z) \cap \Omega} |Du|^2 \rightarrow 0 \text{ as } r \rightarrow 0 \text{ uniformly for all } z \in \partial\Omega;$$

then u is continuous on $\overline{\Omega}$.

Proof. Observe that in case of $N = 2$, $r^{2-N} \int_{B_r(z) \cap \Omega} |Du|^2 = \int_{B_r(z) \cap \Omega} |Du|^2 \rightarrow 0$ uniformly due to the absolute continuity of the integral and $|Du|^2 \in L^1(\Omega)$. Hence it is sufficient to prove the proposition under the assumption that (1.5.7) holds. u is Hölder continuous in the interior (theorem 0.2.3) and so it remains to check that continuity extends up to the boundary. This is a local question so we assume that $\Omega = \Omega_F$ for some Lipschitz continuous F , with Lipschitz norm $\text{Lip}(F) < L$. Furthermore let $z_0 = (z', F(z')) \in \partial\Omega_F$ be fixed.

Consider a generic sequence $x_k = (x'_k, x_{N,k})$ converging to z_0 from the interior. Set $r_k = x_{N,k} - F(x'_k) > 0$ and $\epsilon = \frac{1}{2\sqrt{1+L^2}}$. Then $B_{2\epsilon r_k}(x_k) \subset \Omega_F$ for all k and

$$(1.5.8) \quad r_k^2 \leq 2(x_{N,k} - z_N)^2 + 2(F(z') - F(x'_k))^2 \leq \frac{1}{2\epsilon^2} |x_k - z_0|^2.$$

To show continuity we have to check that $\mathcal{G}(u(x_k), u|_{\partial\Omega_F}(z_0))$ is of order $o(1)$. The triangle inequality and convexity gives

$$\begin{aligned} \frac{1}{3}\mathcal{G}(u(x_k), u|_{\partial\Omega_F}(z_0))^2 &\leq \mathcal{G}(u(x_k), u(x))^2 \\ &+ \mathcal{G}(u(x), u|_{\partial\Omega_F}(x'))^2 + \mathcal{G}(u|_{\partial\Omega_F}(x'), u|_{\partial\Omega_F}(z_0))^2. \end{aligned}$$

Integration in $x \in B_{\epsilon r_k}(x_k)$ gives

$$\begin{aligned} \frac{1}{3}\mathcal{G}(u(x_k), u|_{\partial\Omega_F}(z_0))^2 &\leq \int_{B_{\epsilon r_k}(x_k)} \mathcal{G}(u(x_k), u(x))^2 \\ &+ \int_{B_{\epsilon r_k}(x_k)} \mathcal{G}(u(x), u|_{\partial\Omega_F}(x'))^2 + \int_{B_{\epsilon r_k}(x_k)} \mathcal{G}(u|_{\partial\Omega_F}(x'), u|_{\partial\Omega_F}(z_0))^2. \end{aligned}$$

It is sufficient to check that all integrals are of order $o(1)$.

$$\int_{B_{\epsilon r_k}(x_k)} \mathcal{G}(u|_{\partial\Omega_F}(x'), u|_{\partial\Omega_F}(z_0))^2 \leq \sup_{x \in B_{|x_k - z_0|}(z_0)} \mathcal{G}(u|_{\partial\Omega_F}(x'), u|_{\partial\Omega_F}(z_0))^2 = o(1)$$

where we used (1.5.8) and assumption (a1).

For a fixed k set $\tilde{\Omega} = \{(x', x_N) : x' \in \Omega', F(x') < x_N < G(x')\}$ with $\Omega' = B_{\epsilon r_k}(x'_k) \subset \mathbb{R}^{N-1}$, $G(x') = x_{N,k} + \epsilon r_k$. The trace estimate, Lemma 1.5.3 states

$$\frac{1}{r_k^2} \int_{\tilde{\Omega}} \mathcal{G}(u(x), u|_{\partial\Omega_F}(x'))^2 \leq 4 \int_{\tilde{\Omega}} \frac{\mathcal{G}(u(x), u|_{\partial\Omega_F}(x'))^2}{|x_N - F(x')|^2} \leq 16 \int_{\tilde{\Omega}} |Du|^2;$$

where we used $\frac{1}{r_k} \leq \frac{2}{x_N - F(x')}$ because of $x_N - F(x') = x_N - x_{N,k} + r_k + F(x'_k) - F(x') \leq \epsilon r_k + r_k + L\epsilon r_k \leq 2r_k$. We may combine it with $B_{\epsilon r_k}(x_k) \subset \tilde{\Omega} \subset B_{2r_k}(z_k) \cap \Omega_F$ and assumption (a2) to deduce

$$\int_{B_{\epsilon r_k}(x_k)} \mathcal{G}(u(x), u|_{\partial\Omega_F}(x'))^2 \leq \frac{16}{\omega_N \epsilon^N r_k^{2-N}} \int_{B_{2r_k}(z_k) \cap \Omega_F} |Du|^2 = o(1).$$

Finally the first integral is estimated using the internal Hölder continuity result: since $B_{2\epsilon r_k}(x_k) \subset \Omega_F$ for positive C, β

$$\mathcal{G}(u(x), u(x_k))^2 \leq C \left(\frac{|x - x_k|}{\epsilon r_k} \right)^{2\beta} (\epsilon r_k)^{2-N} \int_{B_{2\epsilon r_k}(x_k)} |Du|^2 \text{ for all } x \in B_{\epsilon r_k}(x_k).$$

Integration in x and $B_{2\epsilon r_k}(x_k) \subset B_{2r_k}(z_k)$ gives

$$\int_{B_{\epsilon r_k}(x_k)} \mathcal{G}(u(x), u(x_k))^2 \leq \frac{C}{(\epsilon r_k)^{N-2}} \int_{B_{2\epsilon r_k}(x_k)} |Du|^2 \leq \frac{C}{\epsilon^{N-2} r_k^{2-N}} \int_{B_{2r_k}(z_k)} |Du|^2;$$

that is of order $o(1)$ by assumption (a2). \square

Remark 1.5.2. $u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ implies that $u|_{\partial\Omega} \in W^{\frac{1}{2},2}(\partial\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ but this is just not sufficient to ensure continuity. $W^{\frac{1}{2},2}(\mathbb{R}) = H^{\frac{1}{2}}(\mathbb{R})$ does not embed into $L^\infty(\mathbb{R})$ but only the slightly smaller space $(H^1(\mathbb{R}), L^2(\mathbb{R}))_{\frac{1}{2},1}$ embeds into $C^0(\mathbb{R})$, compare for instance [23, chapter 25].

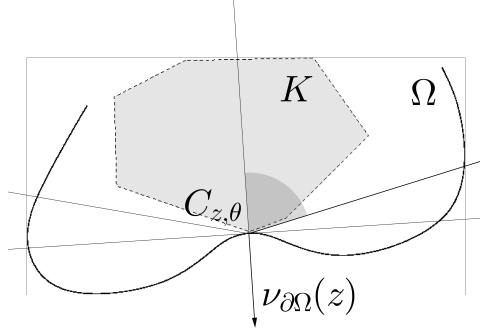
1.5.3. Partial improvement of the Hölder exponent. In the introduction we mentioned already that it would be desirable to extend the optimal Hölder exponent $\frac{1}{Q}$ in the interior up to the boundary. We want to present in this subsection a partial improvement of theorem 1.5.2:

Let $\Omega \subset \mathbb{R}^2$ be a C^1 -regular domain the following holds:

$u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ Dirichlet minimizing with $u|_{\partial\Omega} \in C^{0,\beta}(\partial\Omega)$ for some $\beta > \frac{1}{2}$ then $u \in C^{0,\alpha}(K)$, if $\alpha = \frac{1}{Q}$ for $Q > 2$, $0 < \alpha < \frac{1}{2}$ for $Q = 2$ and if $K \subset \bar{\Omega}$ has the property that it is closed and touches $\partial\Omega$ in at most 1 point z non-tangential.

To every closed set K of this type there is a cone $C_{z,\theta} = \{x \in \mathbb{R}^2: |x| \cos(\theta) < -\langle \nu_{\partial\Omega}(z), x \rangle\}$ for some $0 < \theta < \frac{\pi}{2}$ ($\nu_{\partial\Omega}(z)$ denotes the outward pointing normal to $\partial\Omega$ at z) and a radius $0 < R$ s.t. $K \cap B_R(z) \subset \overline{C_{z,\theta}} \cap B_R(z)$. Shrinking $R > 0$ if necessary we may even assume w.l.o.g. that $C_{z,\theta} \cap B_R(z) \subset \Omega$. This is sketched in the figure.

$K \setminus B_R(z)$ is a compact subset of Ω hence the interior regularity theory holds. It remains to prove regularity for conical subsets $C_{z,\theta} \cap B_R(z)$. The precise statement is:



Corollary 1.5.5. *Let $\frac{1}{2} < s \leq 1$ and $C_\theta = \{x = (x_1, x_2): |x| \cos(\theta) \leq x_2\}$ with $0 < \theta < \frac{\pi}{2}$ (a cone). Under the assumptions*

- (a1) $u \in W^{1,2}(\Omega_F \cap B_1, \mathcal{A}_Q(\mathbb{R}^n))$
Dirichlet minimizing
- (a2) $u|_{\partial\Omega_F} \in W^{s,2}(\Gamma_F, \mathcal{A}_Q(\mathbb{R}^n))$
and for some $0 < \gamma$ there is a
constant $M_u > 0$ s.t.

$$r^{2(s-\gamma)-1} \|u\|_{s,B_r \cap \Gamma_F}^2 \leq M_u^2,$$

then there exists $0 < R < 1$ depending on $u(0)$ and θ s.t., for any $\alpha < \min\{\gamma, \frac{1}{2}\}$ and $\alpha \leq \frac{1}{Q}$ the following holds

- (i) $|Du|$ is an element of the Morrey space $L^{2,2\alpha}(\Omega_F \cap B_{\frac{R}{2}} \cap C_\theta)$, more precisely

$$(1.5.9) \quad r^{-2\alpha} \int_{B_r(x) \cap \Omega_F} |Du|^2 \leq \frac{4}{\delta^{2\alpha}} \left(\int_{B_R \cap \Omega_F} |Du|^2 + \frac{CR^{2(\gamma-\alpha)}}{\gamma-\alpha} M_u^2 \right)$$

where $\delta = \cos(\theta) - \cos(\frac{2\theta+\pi}{4})$;

- (ii) $u \in C^{0,\alpha}(\overline{\Omega_F \cap B_{\frac{R}{2}} \cap C_\theta})$.

Concerning the optimality of the achieved Hölder exponent and assumption (a2) consider the following:

Remark 1.5.3. (a2) is obviously always satisfied for $\gamma = s - \frac{1}{2}$.

(a2) is satisfied for $\gamma > \frac{1}{2}$ and any $s < \gamma$ if $u|_{\Gamma_F} \in C^{0,\gamma}(\Gamma_F)$ as we have seen in lemma 1.4.2. Furthermore this implies that

$$u \in C^{0,\alpha}(\overline{\Omega_F \cap B_R \cap C_\theta}) \text{ with } \alpha = \frac{1}{Q} \text{ for } Q > 2 \text{ and any } \alpha < \frac{1}{2} \text{ for } Q = 2;$$

i.e. the optimal exponent extends on cones up to the boundary.

The proof of the corollary follows similar lines as in the higher dimensional case. We will prove an improved estimate in the spirit of proposition 1.4.3, that will lead eventually to corollary 1.5.5. Before we present this final argument we prove the preliminary lemmas. As in the previous sections: $B_{1+} = B_1 \cap \{x_2 > 0\}$, $\mathcal{S}^1 = \partial B_1$, $\mathcal{S}_+^1 = \mathcal{S}^1 \cap \{x_2 > 0\}$, and $\Gamma_0 = B_1 \cap \{x_2 = 0\}$.

Lemma 1.5.6. *Let $\frac{1}{2} < s \leq 1$ be given, then there is a constant $C = C(s)$ s.t. any single valued harmonic function $f \in W^{1,2}(B_{1+})$ satisfies*

$$(1.5.10) \quad \int_{B_{1+}} |Df|^2 \leq (1+\epsilon) \int_{\mathcal{S}_+^1} |D_\tau f|^2 + \frac{C}{\epsilon} \int_{\Gamma_0} \|f\|_{s,\Gamma_0}^2 \quad \forall \epsilon > 0.$$

Proof. In a first step we show the existence of $C = C(s)$ s.t. any classical single-valued harmonic $h \in W^{1,2}(B_{1+})$ satisfies

$$(1.5.11) \quad \int_{B_{1+}} |Dh|^2 \leq C \left(\int_{S_+^1} |D_\tau h|^2 + \|h\|_{s,\Gamma_0}^2 \right).$$

If $h \notin W^{s,2}(\Gamma_0)$ the RHS is $+\infty$ so there is nothing to check. $G : \overline{B_1} \rightarrow \overline{B_{1+}}$ denotes the bilipschitz map of Lemma C.1. Let $\sum_{k \in \mathbb{Z}} a_k e^{ik\theta}$ be the Fourier series of $h \circ G|_{S^1} = h|_{S^1} \circ G$. Its harmonic extension is then

$$\tilde{h}(r e^{i\theta}) = \sum_{k \in \mathbb{Z}} a_k r^k e^{ik\theta}.$$

h is harmonic, hence minimizing the Dirichlet energy, and $\tilde{h} \circ G^{-1}$ is an admissible competitor, so that

$$\int_{B_{1+}} |Dh|^2 \leq \int_{B_{1+}} |D(\tilde{h} \circ G^{-1})|^2 \leq C \int_{B_1} |D\tilde{h}|^2 = C 2\pi \sum_{k \in \mathbb{Z}} |k| |a_k|^2.$$

For $s = 1$ we estimate (the constant C depends only on the Lipschitz norms of G, G^{-1})

$$\begin{aligned} 2\pi \sum_{k \in \mathbb{Z}} |k| |a_k|^2 &\leq 2\pi \sum_{k \in \mathbb{Z}} k^2 |a_k|^2 = \int_{S_+^1} |D_\tau \tilde{h}|^2 + \int_{S_-^1} |D_\tau \tilde{h}|^2 \\ &\leq C \left(\int_{S_+^1} |D_\tau h|^2 + \int_{\Gamma_0} |D_\tau h|^2 \right). \end{aligned}$$

for $\frac{1}{2} < s < 1$:

(A short auxiliary argument: Lemma A.14 implies the equivalence of the norms $|b_0| + \sum_{k \in \mathbb{Z}} |k|^{2s} |b_k|^2$ and $\|f\|_{L^2(S^1)}^2 + \|f\|_{s,S^1}^2$ for a function $f(\theta) = \sum_{k \in \mathbb{Z}} b_k e^{ik\theta}$. In the case of S^1 this follows more directly. $f(\theta + \tau) - f(\theta) = \sum_{k \in \mathbb{Z}} (e^{ik\tau} - 1) a_k e^{ik\theta}$ and therefore

$$\int_0^{2\pi} |f(\theta + \tau) - f(\theta)|^2 d\theta = \sum_{k \in \mathbb{Z}} 4 \sin^2\left(\frac{k}{2}\tau\right) |a_k|^2.$$

This implies

$$\begin{aligned} \int_{[0,2\pi]^2} \frac{|f(\theta) - f(\varphi)|^2}{|\theta - \varphi|^{1+2s}} d\theta d\varphi &= \int_0^{2\pi} \frac{1}{\tau^{1+2s}} \int_0^{2\pi} |f(\theta + \tau) - f(\theta)|^2 d\theta d\tau \\ &= \sum_{k \in \mathbb{Z}} |a_k|^2 \left(4 \int_0^{2\pi} \frac{\sin^2(\frac{k}{2}\tau)}{\tau^{1+2s}} d\tau \right) = \sum_{k \in \mathbb{Z}} c_k |k|^{2s} |a_k|^2; \end{aligned}$$

where

$$\int_0^{2\pi} \frac{\sin^2(\frac{k}{2}\tau)}{\tau^{1+2s}} d\tau = |k|^{2s} 4^{1-s} \int_0^{k\pi} \frac{\sin^2(\tau)}{\tau^{1+2s}} d\tau = |k|^{2s} c_k$$

and $0 < c_1 \leq c_k \leq c_\infty < \infty$.)

Firstly the auxiliary argument gives

$$2\pi \sum_{k \in \mathbb{Z}} |k| |a_k|^2 \leq 2\pi \sum_{k \in \mathbb{Z}} |k|^{2s} |a_k|^2 \leq C \|\tilde{h}\|_{s,S^1}^2;$$

secondly Corollary A.8 gives

$$\|\tilde{h}\|_{s,S^1}^2 \leq C \left(\|\tilde{h}\|_{s,S^1 \cap \{x_2 > \frac{1}{5}\}}^2 + \|\tilde{h}\|_{s,S^1 \cap \{x_2 < \frac{1}{5}\}}^2 \right);$$

thirdly G is Lipschitz continuous and $G(\mathcal{S}^1 \cap \{x_2 > \frac{1}{5}\}) = \mathcal{S}_+^1, G(\mathcal{S}^1 \cap \{x_2 < \frac{1}{5}\}) = \Gamma_0$ so that

$$\|\tilde{h}\|_{s, \mathcal{S}^1 \cap \{x_2 > \frac{1}{5}\}}^2 + \|\tilde{h}\|_{s, \mathcal{S}^1 \cap \{x_2 < \frac{1}{5}\}}^2 \leq C \left(\|h\|_{s, \mathcal{S}_+^1}^2 + \|h\|_{s, \Gamma_0}^2 \right);$$

finally combining these with the interpolation property $\|\cdot\|_{s, \mathcal{S}_+^1} \leq C \|\cdot\|_{s, \mathcal{S}_+^1}$ we estimate

$$2\pi \sum_{k \in \mathbb{Z}} |k| |a_k|^2 \leq C \left(\int_{\mathcal{S}_+^1} |D_\tau h|^2 + \|h\|_{s, \Gamma_0}^2 \right).$$

Hence (1.5.11) holds.

Now we are able to improve (1.5.11) to (1.5.10). Let f be the harmonic function as assumed. We may assume $f \in W^{s,2}(\Gamma_0)$ otherwise the RHS is $+\infty$ and (1.5.10) holds trivially. Define the linear function

$$l(x_1, x_2) = \frac{f(1, 0) - f(-1, 0)}{2} x_1 + \frac{f(1, 0) + f(-1, 0)}{2}.$$

The same calculations as in lemma 1.4.2 give a constant $C = C(s)$ with

$$\|l\|_{s, \Gamma_0}^2 \leq C \|\text{grad } l\|_\infty = C |f(1, 0) - f(-1, 0)|.$$

We achieved that $f(1, 0) - l(1, 0) = 0 = f(-1, 0) - l(-1, 0)$ and hence the glueing lemma A.7 provides that

$$\tilde{h}(x) = \begin{cases} 0, & \text{if } x \in \mathcal{S}_+^1 \\ f(x) - l(x), & \text{if } x \in \Gamma_0 \end{cases}$$

is an element of $W^{s,2}(\mathcal{S}_+^1 \cup \Gamma_0)$. Hence there is a unique harmonic $h \in W^{1,2}(B_{1+})$ with $h|_{\mathcal{S}_+^1 \cup \Gamma_0} = \tilde{h}$. $g = f - (h + l)$ is harmonic in B_{1+} and satisfies $g(x) = 0$ on Γ_0 . The antisymmetric reflexion

$$\tilde{g}(x_1, x_2) = \begin{cases} g(x_1, x_2), & \text{if } x_2 \geq 0 \\ -g(x_1, -x_2), & \text{if } x_2 \leq 0 \end{cases}$$

is by means of the Schwarz reflexion principle harmonic in B_1 with

$$2 \int_{B_{1+}} |Dg|^2 = \int_{B_1} |D\tilde{g}|^2 \leq \int_{\mathcal{S}_+^1} |D_\tau \tilde{g}|^2 = 2 \int_{\mathcal{S}_+^1} |D_\tau g|^2.$$

Young's inequality for $2\langle D_\tau f, D_\tau l \rangle \leq \epsilon |D_\tau f|^2 + \frac{1}{\epsilon} \|\text{grad } l\|_\infty^2$ gives

$$\begin{aligned} \int_{\mathcal{S}_+^1} |D_\tau g|^2 &\leq (1 + \epsilon) \int_{\mathcal{S}_+^1} |D_\tau f|^2 + (1 + \frac{1}{\epsilon}) \pi \|\text{grad } l\|_\infty^2 \\ &= (1 + \epsilon) \int_{\mathcal{S}_+^1} |D_\tau f|^2 + \frac{C}{\epsilon} \|f\|_{s, \Gamma_0}^2 \end{aligned}$$

where we used $\text{grad } l = \frac{f(1,0)-f(-1,0)}{2}$ and $W^{s,2}(\Gamma_0) \subset C^{0,s-\frac{1}{2}}(\Gamma_0)$. Young's inequality for $2\langle D_i f, D_i(h+l) \rangle \geq -\epsilon |D_i f|^2 - \frac{1}{\epsilon} |D_i(h+l)|^2$ gives

$$\int_{B_{1+}} |Dg|^2 \geq (1 - \epsilon) \int_{B_{1+}} |Df|^2 - \frac{1}{\epsilon} \int_{B_{1+}} |D(h+l)|^2;$$

applying (1.5.11) we may conclude

$$\begin{aligned} \int_{B_{1+}} |D(h+l)|^2 &\leq C \left(\int_{\mathcal{S}_+^1} |D_\tau(h+l)|^2 + \|h+l\|_{s, \Gamma_0}^2 \right) \\ &\leq C \left(\pi \|\text{grad } l\|_\infty^2 + \|f\|_{s, \Gamma_0}^2 \right) \leq C \|f\|_{s, \Gamma_0}^2. \end{aligned}$$

□

Lemma 1.5.6 behaves well under perturbations of B_{1+} , as made quantitative in the following corollary.

Corollary 1.5.7. *Let $\frac{1}{2} < s \leq 1$. There is a constant $C > 0$ s.t. to any $\epsilon > 0$ there is $\epsilon_F = \epsilon_F(\epsilon) > 0$ s.t. any single valued harmonic function $f \in W^{1,2}(\Omega_F \cap B_1)$ satisfies*

$$\int_{\Omega_F \cap B_1} |Df|^2 \leq (1 + \epsilon) \int_{\Omega_F \cap S^1} |D_\tau f|^2 + \frac{C}{\epsilon} \|f\|_{s, \Gamma_F}^2.$$

Proof. This follows as a perturbation of the previous lemma making use of the bilipschitz equivalence of $\Omega_F \cap B_1$ and B_{1+} i.e. fix

$$G_F : \overline{B_{1+}} \rightarrow \overline{\Omega_F \cap B_1}$$

as given by lemma C.2. Hence $\|DG_F - \mathbf{1}\|_\infty, \|DG_F^{-1} - \mathbf{1}\|_\infty < 10 \|\text{grad } F\|_\infty \leq 10\epsilon_F$. Let f as assumed with finite RHS, otherwise there is nothing to prove. $f \circ G_F \in W^{1,2}(B_{1+})$ hence there is a unique harmonic $\tilde{f} \in W^{1,2}(B_{1+})$ with $\tilde{f}|_{S^1 \cup \Gamma_0} = f \circ G_F|_{S^1 \cup \Gamma_0}$. f, \tilde{f} are Dirichlet minimizers on their domains so that

$$\int_{\Omega_F \cap B_1} |Df|^2 \leq \int_{\Omega_F \cap B_1} |D(\tilde{f} \circ G_F^{-1})|^2 \leq (1 + 10\epsilon_F)^4 \int_{B_{1+}} |D\tilde{f}|^2.$$

The previous lemma showed that, for some constant $C > 0$,

$$\begin{aligned} \int_{B_{1+}} |D\tilde{f}|^2 &\leq (1 + \epsilon_1) \int_{S_+^1} |D_\tau \tilde{f}|^2 + \frac{C}{\epsilon_1} \|\tilde{f}\|_{s, \Gamma_0}^2 \\ &\leq (1 + \epsilon_1)(1 + 10\epsilon_F)^3 \int_{S^1 \cap \Omega_F} |D_\tau f|^2 + \frac{C}{\epsilon_1} (1 + 10\epsilon_F)^5 \|f\|_{s, \Gamma_F}^2. \end{aligned}$$

We conclude choosing $\epsilon_1 = \frac{\epsilon}{2}$ and then $\epsilon_F > 0$ sufficient small for $(1 + \frac{\epsilon}{2})(1 + 10\epsilon_F)^7 \leq 1 + \epsilon$. □

We can use the obtained results to get an estimate for Dirichlet minimizers in the spirit of proposition 1.4.3.

Lemma 1.5.8. *For $\frac{1}{2} < s \leq 1$ and $\epsilon > 0$, there is a constant $C = C(s) > 0$ with the property that if (A2) holds with $\epsilon_F = \epsilon_F(\epsilon) > 0$ then*

$$\int_{B_r \cap \Omega_F} |Du|^2 \leq (1 + \epsilon) \int_{\partial B_r \cap \Omega_F} |D_\tau u|^2 + \frac{C}{\epsilon} r^{2s-1} \|u\|_{s, B_r \cap \Omega_F}^2 \quad \forall 0 < r < R_0$$

for any Dirichlet minimizing $u \in W^{1,2}(\Omega_F \cap B_1, \mathcal{A}_Q(\mathbb{R}^n))$ and $R_0 = R_0(u(0)) > 0$.

Proof. As usual we may assume that the RHS is finite. Let $\epsilon_F > 0$ be the constant of the previous corollary 1.5.7 and $\|\text{grad } F\|_{\infty, B_1} < \epsilon_F$.

Suppose $s(u(0)) = 0$ i.e. $u(0) = Q[p]$ for some $p \in \mathbb{R}^n$. Since we assumed the RHS is finite $u \in W^{1,2}(\partial B_r \cap \Omega_F, \mathcal{A}_Q(\mathbb{R}^n))$. Fix for such a radius $t_- < 0 < t_+$ and $-\frac{\pi}{2} < \theta_+ < \theta_- < \frac{3\pi}{2}$ s.t.

$$\partial B_r \cap \Omega_F = \{x_+ = (rt_+, F(rt_+)) = re^{i\theta_+}, x_- = (rt_-, F(rt_-)) = re^{i\theta_-}\}.$$

There is $b = (b_1, \dots, b_Q) \in W^{1,2}([\theta_+, \theta_-], \mathbb{R}^{nQ})$ s.t. $[b(\theta)] = u_{0,r}(e^{i\theta}) = u(re^{-i\theta})$ for $\theta_+ \leq \theta \leq \theta_-$ due to the 1-dim. $W^{1,2}$ -selection criterion [12, proposition 1.2]. There are $a(t) = (a_1, \dots, a_Q) \in W^{s',2}([0, t_+], \mathbb{R}^{nQ})$ and $b(t) = (b_1, \dots, b_Q) \in W^{s',2}([t_-, 0], \mathbb{R}^{nQ})$ for any $s' < s$ with $[a(t)] = u(rt, F(rt))$, $[b(t)] = u(rt, F(rt))$

respectively due to the $W^{s,2}$ -selection, lemma B.4. Permuting a and c if necessary we may assume that $a(t_+) = b(\theta_+)$, $c(t_-) = b(\theta_-)$. We may define

$$g(x) = \begin{cases} a(x_1), & \text{if } rx \in B_r \cap \Gamma_F, x_1 \geq 0 \\ b(\theta), & \text{if } rx = re^{i\theta} \in \partial B_r \cap \Omega_F \\ c(x_1), & \text{if } rx \in B_r \cap \Gamma_F, x_1 \leq 0. \end{cases}$$

$g = (g_1, \dots, g_Q) \in W^{s',2}(\partial(B_1, (\Omega_F)_{0,r}), \mathbb{R}^{nQ})$ as a consequence of the gluing lemma A.7. $[g(x)] = \sum_{i=1}^Q [g_i(x)] = u_{0,r}(x)$ for all $x \in \partial(B_1 \cap (\Omega_F)_{0,r})$. Hence there is $h = (h_1, \dots, h_Q) \in W^{1,2}(B_1 \cap (\Omega_F)_{0,r}, \mathbb{R}^{nQ})$ harmonic with g as boundary values. $[h] = \sum_{i=1}^Q [h_i]$ is a competitor to $u_{0,r}$ so that

$$\int_{B_r \cap \Omega_F} |Du|^2 = \int_{B_1 \cap (\Omega_F)_{0,r}} |Du_{0,r}|^2 \leq \int_{B_1 \cap (\Omega_F)_{0,r}} |D[h]|^2 = \int_{B_1 \cap (\Omega_F)_{0,r}} |Dh|^2.$$

The previous corollary 1.5.7 applies to h since $\|\text{grad } F_{0,r}\|_{\infty, B_1} = \|\text{grad } F\|_{\infty, B_r} < \epsilon_F$. So, we find for a fixed $\frac{1}{2} < s' < s$, e.g. $s' = \frac{1+2s}{4}$,

$$\begin{aligned} \int_{B_1 \cap (\Omega_F)_{0,r}} |Dh|^2 &\leq (1+\epsilon) \int_{S^1 \cap (\Omega_F)_{0,r}} |D_\tau h|^2 + \frac{C}{\epsilon} \|h\|_{s', (\Gamma_F)_{0,r}}^2 \\ &\leq (1+\epsilon)r \int_{\partial B_r \cap \Omega_F} |D_\tau u|^2 + \frac{C}{\epsilon} r^{2s-1} \|u\|_{s, \Omega_F \cap B_r}^2 \end{aligned}$$

considering in the last line $[h(x)] = [g(x)] = u_{0,r}(x)$ for $x \in \partial(B_1 \cap (\Omega_F)_{0,r})$ and $\|h\|_{s', (\Gamma_F)_{0,r}} \leq C \|u_{0,r}\|_{s, (\Gamma_F)_{0,r}} = Cr^{2s-1} \|u\|_{s, \Omega_F \cap B_r}^2$ from the $W^{s,2}$ -selection, lemma B.4.

If $s(u(0)) > 0$, i.e. $u(0) = \sum_{j=1}^J Q_j [p_j]$, $|p_i - p_j| \geq s(u(0))$ for $i \neq j$. Fix $R_0 > 0$ s.t.

$$R_0^{\tilde{\alpha}} [u]_{\tilde{\alpha}, \Omega_F \cap B_{R_0}} < \frac{1}{3} s(u(0))$$

where $[\cdot]_{\tilde{\alpha}, \Omega_F \cap B_{R_0}}$ denotes the Hölder semi-norm on $\Omega_F \cap B_{R_0}$ with exponent $\tilde{\alpha} > 0$ provided by theorem 1.5.2. Hence there are Dirichlet minimizing $u_j \in W^{1,2}(\Omega_F \cap B_{R_0}, \mathcal{A}_{Q_j}(\mathbb{R}^n))$ with

$$(1.5.12) \quad \mathcal{G}(u_j(x), Q_j [p_j]) < \frac{1}{3} s(u(0)) \text{ for all } x \in \Omega_F \cap B_{R_0}.$$

To each u_j the assumption $s(u_j(0)) = 0$ is satisfied. So, by the previous considerations for a.e. $0 < r \leq R_0$

$$\begin{aligned} \int_{B_r \cap \Omega_F} |Du|^2 &= \sum_{j=1}^J \int_{B_r \cap \Omega_F} |Du_j|^2 \\ &\leq \sum_{j=1}^J (1+\epsilon)r \int_{\partial B_r \cap \Omega_F} |D_\tau u_j|^2 + \frac{C}{\epsilon} r^{2s-1} \|u_j\|_{s, \Omega_F \cap B_r}^2 \\ &= (1+\epsilon)r \int_{\partial B_r \cap \Omega_F} |D_\tau u|^2 + \frac{C}{\epsilon} r^{2s-1} \|u\|_{s, \Omega_F \cap B_r}^2 \end{aligned}$$

where we used in the last step that $\mathcal{G}(u(x), u(y))^2 = \sum_{j=1}^J \mathcal{G}(u_j(x), u_j(y))^2$ to to (1.5.12). \square

As theorem 1.4.1 follows from proposition 1.4.3, we can now use lemma 1.5.8 to give the final argument leading to the Hölder estimate of corollary 1.5.5.

Proof of corollary 1.5.5. Let $\alpha > 0$ be given as stated. Fix $\epsilon > 0$ s.t. $1 + \epsilon \leq \frac{1}{2\alpha}$ and $0 < R < 1$ sufficient small s.t.

- (1) $R \leq R_0$ when R_0 is the radius of the previous lemma, 1.5.8;

$$(2) \quad \|\text{grad } F\|_{\infty, B_R \cap \Omega_F} < \cos\left(\frac{2\theta+\pi}{4}\right).$$

(2) ensures that $C_\theta \cap B_R \subset C_{\frac{2\theta+\pi}{4}} \cap B_R \subset \Omega_F \cap B_1$. Following the steps in the proof of theorem 1.4.1 for a.e. $0 < r \leq R$

$$\begin{aligned} -\frac{\partial}{\partial r} r^{-2\alpha} \int_{B_r \cap \Omega_F} |Du|^2 &= -r^{-2\alpha} \int_{\partial B_r \cap \Omega_F} |Du|^2 + 2\alpha r^{-2\alpha-1} \int_{B_r \cap \Omega_F} |Du|^2 \\ &\leq \frac{C}{\epsilon} r^{(2s-1-2\alpha)-1} \|u\|_{s, B_r \cap \Gamma_F}^2 \leq \frac{C}{\epsilon} r^{2(\gamma-\alpha)-1} M_u^2. \end{aligned}$$

Integration in $0 < r \leq R$ gives

$$(1.5.13) \quad r^{-2\alpha} \int_{B_r \cap \Omega_F} |Du|^2 \leq R^{-2\alpha} \int_{B_R \cap \Omega_F} |Du|^2 + \frac{CR^{2(\gamma-\alpha)}}{\gamma-\alpha} M_u^2.$$

By definition of $\delta = \cos(\theta) - \cos(\frac{2\theta+\pi}{4})$, for all $x \in B_{\frac{R}{2}} \cap C_\theta$ we have $B_{\delta|x|}(x) \subset C_{\frac{2\theta+\pi}{4}} \cap B_R$. Let $x \in B_{\frac{R}{2}} \cap C_\theta$ and $0 < r < \frac{R}{2}$ be given, set $r_1 = \max\{r, \delta|x|\}$ and $r_2 = r_1 + |x| \leq \frac{2}{\delta} r_1$. We found

$$\begin{aligned} r^{-2\alpha} \int_{B_r(x) \cap \Omega_F} |Du|^2 &\leq r_1^{-2\alpha} \int_{B_{r_1}(x) \cap \Omega_F} |Du|^2 \leq \frac{2^{2\alpha}}{\delta^{2\alpha}} r_2^{-2\alpha} \int_{B_{r_2}(x) \cap \Omega_F} |Du|^2 \\ &\leq \frac{4}{\delta^{2\alpha}} \left(\int_{B_R \cap \Omega_F} |Du|^2 + \frac{CR^{2(\gamma-\alpha)}}{\gamma-\alpha} M_u^2 \right). \end{aligned}$$

where we applied at first the internal estimate since $\alpha \leq \frac{1}{Q}$ and finally the just established (1.5.13). Having established (i), (ii) follows as indicated in the proof of theorem 1.4.1. \square

APPENDIX A. FRACTIONAL SOBOLEV SPACES

We will restrict our overview to the special case of $W^{s,2} = H^s$ for $0 < s < 1$.

A.1. General facts. At first let us consider the spaces on \mathbb{R}^N , there are several ways to define them:

(a) using Fourier transform:

$$H^s(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) \mid |\xi|^s \mathcal{F}u(\xi) \in L^2(\mathbb{R}^N)\};$$

(b) using real interpolation:

$$W^{s,2}(\mathbb{R}^N) = (W^{1,2}(\mathbb{R}^N), L^2(\mathbb{R}^N))_{1-s,2};$$

(c) using the the Gagliardo semi-norm $\|\cdot\|_{s,\mathbb{R}^N}$

$$W^{s,2}(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \|u\|_{s,\mathbb{R}^N}^2 = \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy < \infty \right\};$$

All of these definitions define the same Banach space as can found for instance in [23]: (a)=(c) corresponds to Lemma 16.3 or Lemma 35.2, (a)=(b) can be found in Lemma 23.1.

We will be mostly interested in the case of an open domain $\Omega \subset \mathbb{R}^N$. In this case several definitions are possible, compare [23, section 34 and section 36]:

(a) as restriction

$$W^{s,2}(\Omega) = \text{space of restrictions of functions in } W^{s,2}(\mathbb{R}^N);$$

(b) using interpolation

$$W^{s,2}(\Omega) = (W^{1,2}(\Omega), L^2(\Omega))_{1-s,2};$$

(c) using the Gagliardo norm

$$W^{s,2}(\Omega) = \{u \in L^2(\Omega) : \|u\|_{s,\Omega}^2 = \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy < \infty\};$$

For Ω with Lipschitz boundary one has the existence of an extension operator that is linear and continuous:

$$E : W^{1,2}(\Omega) \rightarrow W^{1,2}(\mathbb{R}^N);$$

E extends to a continuous linear operator mapping $(W^{1,2}(\Omega), L^2(\Omega))_{1-s,2}$ into $(W^{1,2}(\mathbb{R}^N), L^2(\mathbb{R}^N))_{1-s,2}$; therefore (a) and (b) agree in these cases, compare [23, section 34].

For Lipschitz domains one can show the existence of a linear continuous extension operator $\tilde{E} : L^2(\Omega) \rightarrow L^2(\mathbb{R}^N)$ with $\|\tilde{E}u\|_{s,\mathbb{R}^N} \leq \|u\|_{s,\Omega}$, so that all definitions agree; compare [23, Lemma 36.1].

$W^{1,2}(\mathbb{R}^N)$ is dense in $W^{s,2}(\mathbb{R}^N)$ and $W^{1,2}(\Omega)$ in $W^{s,2}(\Omega)$. Since $C_0^\infty(\mathbb{R}^N)$ is dense in $W^{1,2}(\mathbb{R}^N)$ and $C^\infty(\bar{\Omega})$ in $W^{1,2}(\Omega)$, if Ω is Lipschitz regular, the same holds true for the interpolation spaces $W^{s,2}(\mathbb{R}^N)$ and $W^{s,2}(\Omega)$.

The trace spaces are our main concern. Using the characterisation via the Fourier transform one finds the following, [23, Lemma 16.1]:

For $s > \frac{1}{2}$ functions in $H^s(\mathbb{R}^N)$ have a trace on the hyperplane $x_N = 0$ belonging to $H^{s-\frac{1}{2}}(\mathbb{R}^{N-1})$ and this mapping is surjective.

But our concern is the trace on $\partial\Omega$ which will be a C^1 or Lipschitz manifold. We would like to have a statement as follows: For $s > \frac{1}{2}$ functions in $W^{s,2}(\Omega)$ have a

trace $u|_{\partial\Omega}$ belonging to $W^{s-\frac{1}{2},2}(\partial\Omega)$ and this mapping is surjective.

How can we best describe $W^{s,2}(\partial\Omega)$? The definitions (a),(b),(c) for $W^{s,2}(\Omega)$, $\Omega \subset \mathbb{R}^N$ an open Lipschitz regular domain are all non-local. One can check that all definitions share the following property: Let $U_1, U_2 \subset \Omega$ be an open cover of Ω and $u \in L^2(\Omega)$ satisfies $u|_{U_i} \in W^{s,2}(U_i)$ for $i = 1, 2$ then $u \in W^{s,2}(\Omega)$. We are looking now for an general approach to localize that works for all three definitions. This is desirable to define $W^{s,2}(\partial\Omega)$ for a C^1 - or Lipschitz regular domain $\Omega \subset \mathbb{R}^N$ since Ω has the defining property that locally Ω looks like $\Omega_F = \{x \in \mathbb{R}^N : x_N > F(x')\}$, for a C^1 or Lipschitz continuous function F , where $x' = (x_1, \dots, x_{N-1})$. We would like to reduce our analysis to such a local description. For this aim the following two observations are useful:

- (i) equivalence under bilipschitz transformations;
- (ii) one can "localise" and a "local" description controls the global one.

Concerning (i): let $\psi : \Omega' \rightarrow \Omega$ be bilipschitz, Ω N -dimensional; then we may define a linear operator $u \mapsto \psi^\sharp u = u \circ \psi$ with

$$\begin{aligned} \|\psi^\sharp u\|_{L^2(\Omega')} &\leq Lip(\psi^{-1})^{\frac{N}{2}} \|u\|_{L^2(\Omega)} \\ \|\text{grad}(\psi^\sharp u)\|_{L^2(\Omega')} &= \|D\psi^t \text{grad}(u) \circ \psi\|_{L^2(\Omega')} \leq Lip(\psi) Lip(\psi^{-1})^{\frac{N}{2}} \|\text{grad } u\|_{L^2(\Omega)}; \end{aligned}$$

therefore ψ^\sharp extends to a continuous linear operator on the interpolation spaces $(W^{1,2}(\Omega'), L^2(\Omega'))_{1-s,2} \rightarrow (W^{1,2}(\Omega), L^2(\Omega))_{1-s,2}$.

For the Gagliardo semi-norm, we define the constant $C_\psi = Lip(\psi^{-1})^{2N} Lip(\psi)^{N+2s}$ and use $|x - y| \leq Lip(\psi) |\psi^{-1}(x) - \psi^{-1}(y)|$ with a change of variables to conclude that,

$$\int_{\Omega' \times \Omega'} \frac{|\psi^\sharp u(x) - \psi^\sharp u(y)|^2}{|x - y|^{N+2s}} dy dx \leq C_\psi \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dy dx.$$

Concerning (ii): Interpolation behaves well for finite tensor products in the sense that

$$(A.1) \quad \left(\bigotimes_{i=1}^L E_{0,i}, \bigotimes_{i=1}^L E_{1,i} \right)_{\theta,p} = \bigotimes_{i=1}^L (E_{0,i}, E_{1,i})_{\theta,p}.$$

We will show that below. Assuming (A.1) holds true we can check (ii). Given any finite open cover $\{U_i\}_{i=1,\dots,L}$ of Ω with subordinate partition of unity $(\theta_i)_{i=1,\dots,L}$ we define

$$R : W^{1,2}(\Omega) \rightarrow \bigotimes_{i=1}^L W^{1,2}(U_i) \quad Ru = (u_1, \dots, u_L),$$

where u_i is the restriction of u to U_i , and

$$T : \bigotimes_{i=1}^L W^{1,2}(U_i) \rightarrow W^{1,2}(\Omega) \quad T(u_1, \dots, u_L) = \sum_{i=1}^L \theta_i u_i.$$

Both operators are linear and continuous, because

$$\begin{aligned}
\sum_{i=1}^L \|u_i\|_{L^2(U_i)} &\leq L \|u\|_{L^2(\Omega)} \\
\sum_{i=1}^L \|\text{grad}(u_i)\|_{L^2(U_i)} &\leq L \|\text{grad}(u)\|_{L^2(\Omega)} \\
\left\| \sum_{i=1}^L \theta_i u_i \right\|_{L^2(\Omega)} &\leq \sum_{i=1}^L \|u_i\|_{L^2(U_i)} \\
\left\| \sum_{i=1}^L \text{grad}(\theta_i u_i) \right\|_{L^2(\Omega)} &\leq \sum_{i=1}^L |\text{grad}(\theta_i)|_\infty \|u_i\|_{L^2(U_i)} + |\theta_i|_\infty \|\text{grad}(u_i)\|_{L^2(U_i)}.
\end{aligned}$$

Using (A.1) they extend to linear continuous operators

$$\begin{aligned}
R : W^{s,2}(\Omega) &\rightarrow \bigotimes_{i=1}^L W^{s,2}(U_i) \\
T : \bigotimes_{i=1}^L W^{s,2}(U_i) &\rightarrow W^{s,2}(\Omega).
\end{aligned}$$

By definition $T \circ R = \mathbf{1}_{W^{s,2}(\Omega)}$ since the equality is obvious on $W^{1,2}(\Omega)$. This shows (ii) in the interpolation case.

It remains to check (A.1). Let $\{(E_{0,i}, E_{1,i})\}_{i=1,\dots,L}$ be finitely many tuples of Banach spaces admissible for interpolation. We can consider the interpolation of their tensor product:

$$\begin{aligned}
E_0 &= \bigotimes_{i=1}^L E_{0,i} \text{ equipped with the norm } \|a\|_0 = \sum_i \|a_i\|_{0,i} \\
E_1 &= \bigotimes_{i=1}^L E_{1,i} \text{ equipped with the norm } \|a\|_1 = \sum_i \|a_i\|_{1,i}
\end{aligned}$$

Hence for the K functional in real interpolation we have

$$K(t, a) = \inf_{\substack{a = a_0 + a_1 \Leftrightarrow \\ a_i = a_{0,i} + a_{1,i}; i=1,\dots,L}} \|a_0\|_0 + t \|a_1\|_1 = \sum_{i=1}^L K_i(t, a_i) \geq K_j(t, a_j)$$

and this establishes (A.1) because

$$\begin{aligned}
\frac{1}{L} \sum_{i=1}^L \|t^{-\theta} K_i(t, a_i)\|_{L^p(\mathbb{R}_+; \frac{dt}{t})} &\leq \|t^{-\theta} K(t, a)\|_{L^p(\mathbb{R}_+; \frac{dt}{t})} \\
&\leq \sum_{i=1}^L \|t^{-\theta} K_i(t, a_i)\|_{L^p(\mathbb{R}_+; \frac{dt}{t})}.
\end{aligned}$$

To check (ii) in the case of the Gagliardo semi-norm we have for the restrictions

$$\sum_{i=1}^L \|u_i\|_{s, U_i} \leq L \|u\|_{s, \Omega}.$$

For an arbitrary Lipschitz function f and $\Omega_1 = \Omega \cap \text{supp}(f)$ write

$$\begin{aligned} \int_{\Omega \times \Omega} \frac{|(fu)(x) - (fu)(y)|^2}{|x - y|^{N+2s}} dy dx &= \int_{\Omega_1 \times \Omega_1} \frac{|(fu)(x) - (fu)(y)|^2}{|x - y|^{N+2s}} dy dx \\ &= \int_{\substack{\Omega_1 \times \Omega_1 \\ |x-y| < 1}} \frac{|(fu)(x) - (fu)(y)|^2}{|x - y|^{N+2s}} dy dx + \int_{\substack{\Omega_1 \times \Omega_1 \\ |x-y| \geq 1}} \frac{|(fu)(x) - (fu)(y)|^2}{|x - y|^{N+2s}} dy dx; \end{aligned}$$

for the second integral we have

$$\int_{\substack{\Omega_1 \times \Omega_1 \\ |x-y| \geq 1}} \frac{|(fu)(x) - (fu)(y)|^2}{|x - y|^{N+2s}} dy dx \leq 4|f|_\infty^2 \frac{N\omega_N}{2s} \int_{\Omega_1} |u|^2$$

where we used symmetry in x, y and

$$\int_{\Omega \setminus B_1(x)} \frac{1}{|x - y|^{N+2s}} dy \leq N\omega_N \int_1^\infty r^{-1-2s} dr = \frac{N\omega_N}{2s};$$

for the first integral we have

$$\begin{aligned} &\int_{\substack{\Omega_1 \times \Omega_1 \\ |x-y| < 1}} \frac{|(fu)(x) - (fu)(y)|^2}{|x - y|^{N+2s}} dy dx \\ &\leq 2|f|_\infty^2 \int_{\Omega_1 \times \Omega_1} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dy dx + \text{Lip}(f)^2 \frac{2N\omega_N}{2-2s} \int_{\Omega_1} |u|^2 \end{aligned}$$

where we used $|(fu)(x) - (fu)(y)| \leq |f|_\infty |u(x) - u(y)| + |f(x) - f(y)| |u(x)| \leq |f|_\infty |u(x) - u(y)| + |u(x)| \text{Lip}(f) |x - y|$ and

$$\int_{\Omega \cap B_1(x)} \frac{|x - y|^2}{|x - y|^{N+2s}} \leq N\omega_N \int_0^1 r^{2-2s-1} dr = \frac{N\omega_N}{2-2s}.$$

Hence we got the desired estimate with the constant $C_f = 2|f|_\infty^2 + \frac{2N\omega_N}{s(1-s)} (\text{Lip}(f)^2 + |f|_\infty^2)$

$$\int_{\Omega \times \Omega} \frac{|(fu)(x) - (fu)(y)|^2}{|x - y|^{N+2s}} dy dx \leq C_f \left(\|u\|_{s, \Omega_1}^2 + \|u\|_{L^2(\Omega_1)}^2 \right).$$

Using this estimate we can conclude (ii) in case of using the Gagliardo semi-norm since

$$\left\| \sum_{i=1}^L \theta_i u_i \right\|_{s, \Omega} \leq \sum_{i=1}^L \|\theta_i u_i\|_{s, \Omega} \leq C \left(\sum_{i=1}^L \|u_i\|_{s, U_i} + \|u_i\|_{L^2(U_i)} \right).$$

Due to (ii) it is sufficient to consider the case Ω_F , Furthermore using (i) with the bilipschitz mapping $(x', x_N) \mapsto (x', x_N + F(x'))$ between \mathbb{R}_+^N and Ω_F , it is sufficient to understand \mathbb{R}_+^N . Hence as definition for the spaces on the boundary we may use

$$W^{s,2}(\partial\Omega_F) = \{u(x', x_N - F(x')) : u \in W^{s,2}(\mathbb{R}_+^N)\};$$

for the Gagliardo seminorm we may use as well the global version

$$\|u\|_{s, \partial\Omega}^2 = \int_{\partial\Omega \times \partial\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N-1+2s}} dy dx.$$

Corollary A.1. *For $s > \frac{1}{2}$ functions of $W^{s,2}(\mathbb{R}_+^N)$ have a trace on the hyperplane $x_N = 0$ belonging to $W^{s-\frac{1}{2},2}(\mathbb{R}^{N-1})$ and this linear continuous mapping $|_{\partial\mathbb{R}_+^N}$ is surjective.*

Proof. $u \in W^{s,2}(\mathbb{R}_+^N)$ if and only if the extension

$$Eu(x) = \begin{cases} u(x', x_N), & \text{if } x_N > 0 \\ u(x', -x_N), & \text{if } x_N < 0 \end{cases}$$

is an element of $W^{s,2}(\mathbb{R}^N) = H^s(\mathbb{R}^N)$. Composing this operator with the continuous linear trace operator defined on the whole space using the Fourier transform shows existence. Furthermore it inherits all its properties and hence concludes the proof. \square

The following characterisation for the trace of a function provides a tool to check that a function $u \in W^{s,2}(\Omega)$ can be patched together with a function $v \in W^{s,2}(\mathbb{R}^N \setminus \Omega)$ to a function $U \in W^{s,2}(\mathbb{R}^N)$ if their traces coincide. As introduced before: $\Omega_F = \{x \in \mathbb{R}^N : x_N > F(x')\}$, F Lipschitz continuous

Lemma A.2. *For $u \in W^{s,2}(\Omega_F)$, one has*

$$(A.2) \quad \left\| \frac{u(x', x_N) - u|_{\partial\Omega_F}(x')}{|x_N - F(x')|^s} \right\|_{L^2(\Omega_F)} \leq C \|u\|_{s, \Omega_F}$$

Proof. Using the bilipschitz mapping $(x', x_N) \mapsto (x', x_N - F(x'))$ and $v(x', x_N) = u(x', F(x') + x_N) \in W^{s,2}(\mathbb{R}_+^N)$ together with

$$\int_{\Omega_F} \frac{|u(x', x_N) - u|_{\partial\Omega_F}(x')|^2}{|x_N - F(x')|^{2s}} dx = \int_{\mathbb{R}_+^N} \frac{|u(x', x_N + F(x')) - u|_{\partial\Omega_F}(x')|^2}{|x_N|^{2s}} dx;$$

one has only to consider the case $F = 0$, i.e. \mathbb{R}_+^N .

We may extend u by $u(x', -x_N)$ for $x_N < 0$ to obtain $u \in W^{s,2}(\mathbb{R}^N) = H^s(\mathbb{R}^N)$. We define $v_{x_N}(x') = u(x', x_N)$, then $\mathcal{F}v_{x_N}(\xi') = \int_{\mathbb{R}} e^{2i\pi\xi_N x_N} \mathcal{F}u(\xi', \xi_N) d\xi_N$ and $\mathcal{F}'u|_{\partial\mathbb{R}_+^N}(\xi') = \mathcal{F}v_0(\xi') = \int_{\mathbb{R}} \mathcal{F}u(\xi', \xi_N) d\xi_N$; hence by Cauchy inequality

$$\begin{aligned} |\mathcal{F}v_{x_N}(\xi') - \mathcal{F}v_0(\xi')|^2 &= \left(\int_{\mathbb{R}} (e^{2i\pi\xi_N x_N} - 1) \mathcal{F}u(\xi', \xi_N) d\xi_N \right)^2 \\ &\leq 4 \left(\int_{\mathbb{R}} \frac{|\sin(\pi\xi_N x_N)|}{|\xi_N x_N|^\alpha} x_N d\xi_N \right) x_N^{\alpha-1} \left(\int_{\mathbb{R}} |\sin(\pi\xi_N x_N)| |\xi_N|^\alpha |\mathcal{F}u|^2(\xi', \xi_N) d\xi_N \right); \end{aligned}$$

Multiply this by $|x_N|^{-2s}$ and integrate in x_N to conclude

$$\begin{aligned} &\int_{\mathbb{R}} |x_N|^{-2s} |\mathcal{F}v_{x_N}(\xi') - \mathcal{F}v_0(\xi')|^2 dx_N \\ &\leq 4C(\alpha) \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{|\sin(\pi\xi_N x_N)|}{|\xi_N x_N|^{1+2s-\alpha}} |\xi_N| dx_N \right) |\xi_N|^{2s} |\mathcal{F}u|^2(\xi', \xi_N) d\xi_N \\ &= 4C(\alpha)^2 \int_{\mathbb{R}} |\xi_N|^{2s} |\mathcal{F}u|^2(\xi', \xi_N) d\xi_N \end{aligned}$$

where $C(\alpha) = \int_{\mathbb{R}} \frac{\sin(\pi t)}{|t|^\alpha} dt < \infty$ for $\alpha = 1 + 2s - \alpha$ (note that $1 < \frac{1}{2} + s = \alpha < 2$). This gives the desired result by integrating in ξ' , since

$$\int_{\mathbb{R}^N} \frac{|u(x', x_N) - u|_{\partial\Omega_F}(x')|^2}{|x_N|^{2s}} dx = \int_{\mathbb{R}} |x_N|^{-2s} \int_{\mathbb{R}^{N-1}} |\mathcal{F}v_{x_N}(\xi') - \mathcal{F}v_0(\xi')|^2 d\xi' dx_N.$$

\square

For $s = 1$ compare lemma 1.5.3, that corresponds to [23, Lemma 13.5]. We can conclude the following corollary

Corollary A.3. *$v \in L^2(\mathbb{R}^{N-1})$ is the trace of u (and so in $W^{s-\frac{1}{2},2}(\mathbb{R}^{N-1})$) if*

$$(A.3) \quad \left\| \frac{u(x', x_N) - v(x')}{|x_N - F(x')|^s} \right\|_{L^2(\Omega_F)} < \infty$$

Proof.

$$\begin{aligned}
& \int_{\mathbb{R}^{N-1}} |v(x') - u|_{\partial\Omega_F}(x')|^2 dx' \leq 2\epsilon^{2s} \frac{1}{\epsilon} \int_0^\epsilon \int_{\mathbb{R}^{N-1}} \frac{|v(x') - u(x', F(x') + x_N)|^2}{|x_N|^{2s}} \\
& + 2\epsilon^{2s} \frac{1}{\epsilon} \int_0^\epsilon \int_{\mathbb{R}^{N-1}} \frac{|u(x', F(x') + x_N) - u|_{\partial\Omega_F}(x')|^2}{|x_N|^{2s}} dx' dx_N \\
& \leq 2\epsilon^{2s-1} \left(\left\| \frac{u(x', x_N) - v(x')}{|x_N - F(x')|^s} \right\|_{L^2(\Omega_F)}^2 + \left\| \frac{u(x', x_N) - u|_{\partial\Omega_F}(x')}{|x_N - F(x')|^s} \right\|_{L^2(\Omega_F)}^2 \right);
\end{aligned}$$

converging to 0 as $\epsilon \rightarrow 0$ hence $v = u|_{\partial\Omega_F}$. \square

Corollary A.4. *Let $u \in W^{s,2}(\Omega_F)$ and $v \in W^{s,2}(\mathbb{R}^N \setminus \Omega_F)$ for $s > \frac{1}{2}$ satisfying $u|_{\partial\Omega_F} = v|_{\partial\Omega_F}$ then*

$$(A.4) \quad U(x) = \begin{cases} u(x), & \text{if } x \in \Omega_F \\ v(x), & \text{if } x \in \mathbb{R}^N \setminus \Omega_F \end{cases}$$

defines an element in $W^{s,2}(\mathbb{R}^N)$ satisfying

$$(A.5) \quad \|U\|_{s,\mathbb{R}^N} \leq C (\|u\|_{s,\Omega_F} + \|v\|_{s,\mathbb{R}^N \setminus \Omega_F})$$

Proof. As before using the bilipschitz mapping $(x', x_N) \mapsto (x', x_N - F(x'))$ one has only to consider the case $F = 0$; then

$$\begin{aligned}
& \|U\|_{L^2(\mathbb{R}^N)}^2 = \|u\|_{L^2(\mathbb{R}_+^N)}^2 + \|v\|_{L^2(\mathbb{R}_-^N)}^2 \\
& \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|U(x) - U(y)|^2}{|x - y|^{N+2s}} dy dx = 2 \int_{\mathbb{R}_+^N \times \mathbb{R}_-^N} \frac{|u(x) - v(y)|^2}{|x - y|^{N+2s}} dy dx \\
& + \int_{\mathbb{R}_+^N \times \mathbb{R}_+^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dy dx + \int_{\mathbb{R}_-^N \times \mathbb{R}_-^N} \frac{|v(x) - v(y)|^2}{|x - y|^{N+2s}} dy dx.
\end{aligned}$$

The first two summands are obviously bounded and the third is bounded because

$$(A.6) \quad \int_{\mathbb{R}_+^N \times \mathbb{R}_-^N} \frac{|u(x) - v(y)|^2}{|x - y|^{N+2s}} dy dx \leq 3 \int_{\mathbb{R}_+^N \times \mathbb{R}_-^N} \frac{|u|_{\partial\Omega_F}(x') - v|_{\partial\Omega_F}(y')|^2}{|x - y|^{N+2s}} dy dx$$

$$(A.7) \quad + 3 \int_{\mathbb{R}_+^N \times \mathbb{R}_-^N} \frac{|u(x) - u|_{\partial\Omega_F}(x')|^2}{|x - y|^{N+2s}} dy dx + 3 \int_{\mathbb{R}_+^N \times \mathbb{R}_-^N} \frac{|v|_{\partial\Omega_F}(y') - v(y)|^2}{|x - y|^{N+2s}} dy dx.$$

For the first integral, (A.6), we have

$$\begin{aligned}
& \int_{\mathbb{R}_+^N \times \mathbb{R}_-^N} \frac{|u|_{\partial\Omega_F}(x') - v|_{\partial\Omega_F}(y')|^2}{|x - y|^{N+2s}} dy dx \\
& \leq C_1 \int_{\mathbb{R}^{N-1} \times \mathbb{R}^{N-1}} \frac{|u|_{\partial\Omega_F}(x') - v|_{\partial\Omega_F}(y')|^2}{|x' - y'|^{N-2+2s}} dy' dx' \leq C \|u\|_{s,\mathbb{R}_+^N}^2,
\end{aligned}$$

where we used firstly

$$\int_{\mathbb{R}_+ \times \mathbb{R}_-} \frac{1}{|x - y|^{N+2s}} dx_N dy_N = \int_{\mathbb{R}_+ \times \mathbb{R}_+} \frac{(1 + (t + \tau)^2)^{-\frac{N}{2}-s}}{|x' - y'|^{N-2+2s}} d\tau dt = \frac{C_1}{|x - y|^{N-2+2s}}$$

by means of the change of variables $x_N = |x' - y'|t, y_N = -|x' - y'|\tau$ and then $u|_{\partial\Omega_F} = v|_{\partial\Omega_F}$ together with the continuity of the trace operator $|_{\partial\Omega_F} : W^{s,2}(\mathbb{R}_+^N) \rightarrow W^{s-\frac{1}{2},2}(\mathbb{R}^{N-1})$, compare [23, lemma 16.1, lemma 16.3].

For the second and third integral, (A.7), we proceed equivalently. For instance for the the second

$$\int_{\mathbb{R}_+^N \times \mathbb{R}_-^N} \frac{|u(x) - u|_{\partial \mathbb{R}_+^N}(x')|^2}{|x - y|^{N+2s}} dy dx \leq C_2 \int_{\mathbb{R}_+^N} \frac{|u(x', x_N) - u|_{\partial \Omega_F}(x')|^2}{|x_N|^{2s}} dx \leq C \|u\|_{s, \mathbb{R}_+^N}^2$$

where we used

$$\int_{\mathbb{R}_-^N} \frac{1}{|x - y|^{N+2s}} dy = x_N^{-2s} \int_{\mathbb{R}_+^N} \frac{1}{|z + e_N|^{N+2s}} dz = x_N^{-2s} C_2.$$

by means of the change of variables $(y', y_N) = (x' - x_N z', -x_N z_N)$, $x_N > 0$ and afterwards we apply lemma A.2.

The constants C_1, C_2 are indeed finite since $(t + \tau)^2 \geq t^2 + \tau^2$

$$C_1 \leq \int_0^\infty \int_0^{\frac{\pi}{2}} \frac{r dr d\theta}{(1 + r^2)^{\frac{N}{2}+s}} = \frac{\pi}{2N - 4 + 4s}$$

$$C_2 \leq \int_{\mathbb{R}^N \setminus B_1(-e_N)} \frac{1}{|z + e_N|^{N+2s}} dz = \frac{N\omega_N}{2s}.$$

□

A further nice consequence is the following characterisation of $W_0^{s,2}(\Omega)$, defined as the closure of $C_c^\infty(\Omega)$ in $W^{s,2}(\mathbb{R}^N)$. The "classical" case, $s = 1$, is considered in [23, Lemma 13.6].

Corollary A.5. *If F is Lipschitz continuous and $s > \frac{1}{2}$ then $W_0^{s,2}(\Omega_F)$ is the subspace of $u \in W^{s,2}(\Omega_F)$ satisfying $u|_{\partial \Omega_F} = 0$.*

Proof. If $u \in W_0^{s,2}(\Omega_F)$ there exists a sequence $u_n \in C_c^\infty(\Omega_F)$ s.t. $u_n \rightarrow u$ in $W^{s,2}(\Omega_F)$; as $|_{\partial \Omega_F}$ is a continuous operator on $W^{s,2}(\Omega_F)$ we have $0 = u_n|_{\partial \Omega_F} \rightarrow u|_{\partial \Omega_F}$ in $L^2(\mathbb{R}^{N-1})$.

We may extend u by 0 outside of Ω_F and denote the extension by U . The corollary above shows that $U \in W^{s,2}(\mathbb{R}^N)$. One chooses $0 \leq \theta \leq 1 \in C_c^\infty(\mathbb{R}^N)$ s.t. $\theta(x) = 1$ for $|x| < 1$. One approaches U by the sequence $u_n(x', x_N) = U(x, x_N - \frac{1}{n})\theta(\frac{x}{n}) \in W^{s,2}(\mathbb{R}^N)$. u_n converges to U by Lebesgue dominated convergence. The support of these u_n is compactly supported within Ω_F . Finally regularise u_n by convolution. □

Using interpolation theory there is an elegant way to obtain a statement on compact embeddings:

Lemma A.6. *If $\Omega \subset \mathbb{R}^N$ and bounded, then the injection of $W_0^{s,2}(\Omega)$ into $L^2(\Omega)$ is compact.*

Proof. We have to show that for a bounded sequence $u_n \in W_0^{s,2}(\Omega)$, there is a subsequence converging strongly in $L^2(\Omega)$. To do so it is sufficient to check that for every $\epsilon > 0$ there is a compact subset K_ϵ of $L^2(\Omega)$ s.t. we can decompose $u_n = v_{n,\epsilon} + w_{n,\epsilon}$ with $\|w_{n,\epsilon}\|_{L^2(\Omega)} \leq \epsilon C$ and $v_{n,\epsilon} \in K_\epsilon$ for all n .

Firstly we may extend each u_n by 0 outside of Ω . For a special smoothing sequence $\rho_\epsilon(x) = \frac{1}{\epsilon^N} \rho_0(\frac{x}{\epsilon})$ with ρ_0 radial we can consider the linear operators $u \mapsto u - \rho_\epsilon \star u$. For them we clearly have

$$\|u - \rho_\epsilon \star u\|_{L^2(\mathbb{R}^N)} \leq 2 \|u\|_{L^2(\mathbb{R}^N)}$$

$$\|u - \rho_\epsilon \star u\|_{L^2(\mathbb{R}^N)} \leq \int_{\mathbb{R}^N} \rho_\epsilon(y) \|u(\cdot) - u(\cdot - y)\|_{L^2(\mathbb{R}^N)} dy$$

$$\leq \|\text{grad}(u)\|_{L^2(\mathbb{R}^N)} \int_{\mathbb{R}^N} |y| \rho_\epsilon(y) dy \leq A\epsilon \|\text{grad}(u)\|_{L^2(\mathbb{R}^N)}.$$

$(1 - \rho_\epsilon \star)$ extends to a continuous linear operator on $W^{s,2}(\mathbb{R}^N)$. It therefore satisfies $\|u - \rho_\epsilon \star u\|_{L^2(\mathbb{R}^N)} \leq 2^{1-s} A^s \epsilon^s \|u\|_{W^{s,2}(\mathbb{R}^N)}$. The choice $w_{n,\epsilon} = u_n - \rho_\epsilon \star u_n$ has $\|w_{n,\epsilon}\|_{L^2(\mathbb{R}^N)} \leq C \epsilon^s$ for all n and since $\left\| \frac{\partial \rho_\epsilon \star u_n}{\partial x_j} \right\|_{L^\infty(\mathbb{R}^N)} \leq \left\| \frac{\partial \rho_\epsilon}{\partial x_j} \right\|_{L^2(\mathbb{R}^N)} \|u_n\|_{L^2(\mathbb{R}^N)}$, the sequence $v_{n,\epsilon}$ stays in a bounded set of Lipschitz functions and keeps their support in a fixed compact set of \mathbb{R}^N . The Arzelà-Ascoli theorem provides a subsequence converging strongly in L^∞ and hence L^2 , concluding the statement. \square

The existence of a continuous linear extension operator $E : W^{s,2}(\Omega) \rightarrow W^{s,2}(\mathbb{R}^N)$ for Lipschitz regular domains extends the result to bounded domains i.e. the injection of $W^{s,2}(\Omega)$ into $L^2(\Omega)$ is compact for $\Omega \subset \mathbb{R}^N$ bounded and Lipschitz regular.

As usual the compact embedding can be used to prove Poincaré inequalities:

Lemma A.7. *For a bounded, Lipschitz regular domain $\Omega \subset \mathbb{R}^N$ and $0 < s \leq 1$ there is a constant C_1 s.t. for each $u \in W^{s,2}(\Omega)$*

$$(A.8) \quad \left\| u - \oint_{\Omega} u \right\|_{L^2(\Omega)} \leq C_1 \|u\|_{s,\Omega};$$

for $\frac{1}{2} < s \leq 1$ there is a constant C_2 s.t. for each $u \in W^{s,2}(\Omega)$

$$(A.9) \quad \left\| u - \oint_{\partial\Omega} u|_{\partial\Omega} \right\|_{L^2(\Omega)} \leq C_2 \|u\|_{s,\Omega}.$$

Proof. Both proofs are along the same lines. For the second we need the continuity of the trace operator $|_{\partial\Omega}$ and so $s > \frac{1}{2}$. Nonetheless we will only present the second case and it will be obvious how to argue in the first. We argue by contradiction; so we assume that there exists a sequence $u_k \in W^{s,2}(\Omega)$ with

$$\left\| u_k - \oint_{\partial\Omega} u_k|_{\partial\Omega} \right\|_{L^2(\Omega)} > k \|u_k\|_{s,\Omega}.$$

Normalising via

$$v_k = \frac{u_k - \oint_{\partial\Omega} u_k|_{\partial\Omega}}{\|u_k - \oint_{\partial\Omega} u_k|_{\partial\Omega}\|_{L^2(\Omega)}} \text{ for all } k$$

we may assume that $\|v_k\|_{L^2(\Omega)} = 1$, $\oint_{\partial\Omega} v_k|_{\partial\Omega} = 0$ and by assumption $\|v_k\|_{s,\Omega} < \frac{1}{k}$ for all k . In particular the sequence stays in a fixed bounded set of $W^{s,2}(\Omega)$. We may pass to a subsequence $v_{k'}$ converging strongly in $L^2(\Omega)$ to a function $v \in L^2(\Omega)$, due to the just obtained compact embedding of $W^{s,2}(\Omega)$ into $L^2(\Omega)$. v needs to be constant since $\|v_k\|_{s,\Omega} < \frac{1}{k}$. Thus $v_{k'} \rightarrow v$ strongly in $W^{s,2}(\Omega)$. The continuity of the trace operator provides

$$\oint_{\partial\Omega} v|_{\partial\Omega} = \lim_{k' \rightarrow \infty} \oint_{\partial\Omega} v_{k'}|_{\partial\Omega} = 0.$$

This contradicts $\|v\|_{L^2(\Omega)} = 1$ because $v = \text{const.}$ implies $v|_{\partial\Omega} = \text{const.} = 0$. \square

For our purpose a particular version of corollary A.4 is needed:

Corollary A.8. *To any given $-1 < a < 1$ and $\frac{1}{2} < s \leq 1$ there is a constant $C >$ with the property, that if $u \in W^{s,2}(\mathcal{S}^{N-1} \cap \{x_N > a\})$, $v \in W^{s,2}(\mathcal{S}^{N-1} \cap \{x_N < a\})$ with $u|_{\mathcal{S}^{N-1} \cap \{x_N = a\}} = v|_{\mathcal{S}^{N-1} \cap \{x_N = a\}}$ then*

$$(A.10) \quad U(x) = \begin{cases} u(x), & \text{if } x \in \mathcal{S}^{N-1}, x_N > a \\ v(x), & \text{if } x \in \mathcal{S}^{N-1}, x_N < a \end{cases}$$

defines an element in $W^{s,2}(\mathcal{S}^{N-1})$ satisfying

$$(A.11) \quad \|U\|_{s,\mathcal{S}^{N-1}} \leq C (\|u\|_{s,\mathcal{S}^{N-1} \cap \{x_N > a\}} + \|v\|_{s,\mathcal{S}^{N-1} \cap \{x_N < a\}})$$

Proof. We can apply corollary A.4 locally using a partition of unity $\{\theta_i\}_{i=1}^L$ subordinate to a coordinated atlas $(U_i, \varphi_i)_{i=1, \dots, L}$. More detailed, we may choose a smooth atlas $(U_i, \varphi_i)_{i=1, \dots, L}$ with the additional property that every chart $\varphi_i: U_i \subset \mathcal{S}^{N-1} \rightarrow V_i \subset \mathbb{R}^{N-1}$ satisfies $\varphi_i(U_i \cap \{x_N \geq a\}) = V_i \cap \{y_{N-1} \geq a\}$. We may now apply corollary A.4 to each pair $u|_{U_i \circ \varphi_i^{-1}}, v|_{U_i \circ \varphi_i^{-1}}$ and obtain functions $U_i \in W^{s,2}(V_i)$. Using a subordinated partition of unity $\{\theta_i\}_{i=1}^L$, the function $U(x) = \sum_{i=1}^L \theta_i(x) U_i \circ \varphi_i(x)$ agrees by construction with u on $S^+ = \mathcal{S}^{N-1} \cap \{x_N > a\}$ and with v on $S^- = \mathcal{S}^{N-1} \cap \{x_N < a\}$. Furthermore it satisfies for a constant $C > 0$

$$\|U\|_{s, \mathcal{S}^{N-1}} \leq \|U\|_{W^{s,2}(\mathcal{S}^{N-1})} \leq C \left(\|u\|_{W^{s,2}(S^+)} + \|v\|_{W^{s,2}(S^-)} \right).$$

because every U_i does. To pass to the desired inequality (A.11) we proceed as follows: Given u, v satisfying the assumption, we can apply the above construction to

$$\tilde{u} = u - \oint_{\partial S^+} u|_{\partial S^+}, \quad \tilde{v} = v - \oint_{\partial S^-} v|_{\partial S^-},$$

because \tilde{u}, \tilde{v} still satisfy the assumptions as a consequence of $u|_{\partial S^+} = v|_{\partial S^-}$. We obtain \tilde{U} and U with $\tilde{U} = U - \oint_{\partial S^+} u|_{\partial S^+}$. We can now conclude (A.11) by applying the Poincaré inequality (A.9), since

$$\begin{aligned} \|\tilde{U}\|_{s, \mathcal{S}^{N-1}} &= \|U\|_{s, \mathcal{S}^{N-1}} \\ \|\tilde{u}\|_{W^{s,2}(S^+)} &= \left\| u - \oint_{\partial S^+} u|_{\partial S^+} \right\|_{L^2(S^+)} + \|u\|_{s, S^+} \leq C \|u\|_{s, S^+} \\ \|\tilde{v}\|_{W^{s,2}(S^-)} &= \left\| v - \oint_{\partial S^-} v|_{\partial S^-} \right\|_{L^2(S^-)} + \|v\|_{s, S^-} \leq C \|v\|_{s, S^-}. \end{aligned}$$

□

A.2. Interpolation for fractional Sobolev functions. Commonly one can use a version of the Luckhaus' lemma to interpolate between two functions on the sphere. If an L^∞ -estimate is not needed it states:

To any $0 < \epsilon < \frac{1}{2}$ and $u, v \in W^{1,2}(\mathcal{S}^{N-1})$ there is $w \in W^{1,2}(B_1 \setminus B_{(1-\epsilon)})$ with $w(x) = u(x)$ and $w((1-\epsilon)x) = v(x)$ for all $x \in \mathcal{S}^{N-1}$, satisfying

$$(A.12) \quad \int_{B_1 \setminus B_{1-\epsilon}} |Dw|^2 \leq 2\epsilon \int_{\mathcal{S}^{N-1}} |D_\tau u|^2 + |D_\tau v|^2 + \frac{1}{\epsilon} \int_{\mathcal{S}^{N-1}} |u - v|^2$$

Define a linear interpolation on the cylinder $\mathcal{S}^{N-1} \times [0, \epsilon]$ by

$$\tilde{w}(y, t) = \left(1 - \frac{t}{\epsilon}\right) u(y) + \left(\frac{t}{\epsilon}\right) v(y) \text{ for } y \in \mathcal{S}^{N-1}, t \in [0, \epsilon]$$

and then making use of polar coordinates $x = ry, r \in [1-\epsilon, 1], y \in \mathcal{S}^{N-1}$ the annulus $A_{1,1-\epsilon} = B_1 \setminus B_{1-\epsilon}$ is close to the cylinder i.e.

$$w(ry) = \tilde{w}(y, 1-r) \text{ for } r \in [1-\epsilon, 1], y \in \mathcal{S}^{N-1} \text{ i.e. } ry \in A_{1,1-\epsilon}.$$

One checks that w defined in that way satisfies (A.12).

Our extension of this result to "boundary" functions in a fractional Sobolev space is:

Lemma A.9. *Let $\frac{1}{2} < s < 1$ and $\epsilon > 0$ be given then there exists $R_\epsilon > 0$ with the property: for any $R_\epsilon \leq R < 1$ there is $C = C(\epsilon, R)$ s.t. given $u, v \in W^{s,2}(\mathcal{S}^{N-1})$*

one can find $w \in W^{1,2}(A_{1,R})$ on the annulus $A_{1,R} = B_1 \setminus B_R$ with $w(x) = u(x)$ and $w(Rx) = v(x)$ for $x \in \mathcal{S}^{N-1}$ that satisfies

$$(A.13) \quad \int_{A_{1,R}} |Dw|^2 \leq \epsilon \left(\|u\|_{s, \mathcal{S}^{N-1}}^2 + \|v\|_{s, \mathcal{S}^{N-1}}^2 \right) + C \|u - v\|_{\mathcal{S}^{N-1}}^2.$$

Our proof uses heavily the theory of homogenous harmonic polynomials. This is not a surprise since they build, together with their Kelvin transforms, a natural basis for solving the Dirichlet problem on an annulus. As a reference for classical results one may consult [3, chapter 5].

We will use the same notation introduced there:

- $\mathcal{P}_m(\mathbb{R}^N)$ denotes the complex vector space of all homogeneous polynomials on \mathbb{R}^N of degree m ;
- $\mathcal{H}_m(\mathbb{R}^N) \subset \mathcal{P}_m(\mathbb{R}^N)$ the subspace of all harmonic homogeneous polynomials of degree m .

We want to emphasise that we do not equip $\mathcal{P}_m(\mathbb{R}^N)$ and $\mathcal{H}_m(\mathbb{R}^N)$ with specific norms or inner products.

Furthermore we need the Kelvin transform for a map $u : \Omega \subset \mathbb{R}^N \setminus \{0\}$

$$(A.14) \quad K[u] = |x|^{2-N} u \left(\frac{x}{|x|^2} \right) \text{ for } x \in \Omega^* = \left\{ x : \frac{x}{|x|^2} \in \Omega \right\}.$$

A key feature of the Kelvin transform is $\Delta(K[u]) = K[|x|^4 \Delta u]$, compare [3, Proposition 4.6]. Hence the Kelvin transform is a homeomorphism on harmonic functions, [3, Theorem 4.7]. Furthermore for $p \in \mathcal{P}_m(\mathbb{R}^N)$ we have the simple formula $K[p](x) = \frac{p(x)}{|x|^{N+2m-2}}$. $K[p]$ is therefore homogeneous of degree $2 - N - m$.

The proof of lemma A.9 splits into two parts.

In the first we characterise $W^{s,2}(\mathcal{S}^{N-1})$ using a Fourier decomposition into harmonic homogeneous polynomials. In the second we use this characterisation to estimate the solution of the Dirichlet problem on the annulus $A_{1,R} = B_1 \setminus B_R$.

Recall the classical theorem, e.g. [3, Theorem 5.7]

Theorem A.10. *Every $p \in \mathcal{P}_m(\mathbb{R}^N)$ can be uniquely written in the form*

$$p = p_m + |x|^2 p_{m-2} + \cdots + |x|^{2k} p_{m-2k},$$

where $k = \lfloor \frac{m}{2} \rfloor$ and each $p_n \in \mathcal{H}_n(\mathbb{R}^N)$.

Lemma A.11. *If $p \in \mathcal{H}_m(\mathbb{R}^N)$ and q is a polynomial with strictly less degree then*

$$(A.15) \quad \int_{\mathcal{S}^{N-1}} pq = 0 = \int_{\mathcal{S}^{N-1}} D_\tau p \cdot D_\tau q$$

$$(D_\tau p \cdot D_\tau q = Dp \cdot Dq - \frac{\partial p}{\partial r} \frac{\partial q}{\partial r} = \sum_{i=1}^N \frac{\partial p}{\partial x_i} \frac{\partial q}{\partial x_i} - \frac{\partial p}{\partial r} \frac{\partial q}{\partial r})$$

If $p, q \in \mathcal{H}_m(\mathbb{R}^N)$ then

$$(A.16) \quad \begin{aligned} m(N-2+2m) \int_{\mathcal{S}^{N-1}} pq &= \int_{\mathcal{S}^{N-1}} Dp \cdot Dq \\ &= \int_{\mathcal{S}^{N-1}} D_\tau p \cdot D_\tau q + m^2 \int_{\mathcal{S}^{N-1}} \frac{\partial p}{\partial r} \frac{\partial q}{\partial r}. \end{aligned}$$

Proof. By linearity and the decomposition of theorem A.10 we may assume that $q \in \mathcal{H}_n(\mathbb{R}^N)$ for some $n < m$. Recall that if $u \in C^1$ is homogenous of degree λ , it satisfies the Euler formula $|x| \frac{\partial u}{\partial r}(x) = Du(x) \cdot x = \lambda u(x)$. Furthermore observe

that $\frac{\partial p}{\partial x_i} \in \mathcal{H}_{m-1}(\mathbb{R}^N)$ and $\frac{\partial q}{\partial x_i} \in \mathcal{H}_{n-1}(\mathbb{R}^N)$ for any $i = 1, \dots, N$. Hence

$$\begin{aligned} n \int_{\mathcal{S}^{N-1}} pq &= \int_{\mathcal{S}^{N-1}} p \frac{\partial q}{\partial r} = \int_{\mathcal{S}^{N-1}} \frac{\partial p}{\partial r} q + \int_{B_1} p \Delta q - \Delta p q \\ &= m \int_{\mathcal{S}^{N-1}} pq; \\ \int_{\mathcal{S}^{N-1}} D_\tau p \cdot D_\tau q &= \int_{\mathcal{S}^{N-1}} D_\tau p \cdot D_\tau q + nm pq = \int_{\mathcal{S}^{N-1}} D_\tau p \cdot D_\tau q + \frac{\partial p}{\partial r} \frac{\partial q}{\partial r} \\ &= \sum_{i=1}^N \int_{\mathcal{S}^{N-1}} \frac{\partial p}{\partial x_i} \frac{\partial q}{\partial x_i} = 0; \end{aligned}$$

where we applied the (just obtained) orthogonality of $\mathcal{H}_m(\mathbb{R}^N)$ to $\mathcal{H}_n(\mathbb{R}^N)$ for $m \neq n$.

To show (A.16) observe that pq is homogenous of degree $2m$ hence

$$\begin{aligned} m(N-2+2m) \int_{\mathcal{S}^{N-1}} pq &= \frac{1}{2}(N-2+2m) \int_{\mathcal{S}^{N-1}} \frac{\partial(pq)}{\partial r} \\ &= (N-2+2m) \int_{B_1} Dp \cdot Dq = (N-2+2m) \int_0^1 \int_{\mathcal{S}^{N-1}} (Dp \cdot Dq)(rx) r^{N-1} dr \\ &= (N-2+2m) \int_0^1 r^{2m-2+N-1} dr \int_{\mathcal{S}^{N-1}} Dp \cdot Dq = \int_{\mathcal{S}^{N-1}} Dp \cdot Dq \\ &= \int_{\mathcal{S}^{N-1}} D_\tau p \cdot D_\tau q + \frac{\partial p}{\partial r} \frac{\partial q}{\partial r} = \int_{\mathcal{S}^{N-1}} D_\tau p \cdot D_\tau q + m^2 \int_{\mathcal{S}^{N-1}} pq. \end{aligned}$$

□

On the base of some Hilbert space theory we recover the following classical result and a small extension, compare e.g. [3, Theorem 5.12]:

Theorem A.12.

$$\begin{aligned} (A.17) \quad L^2(\mathcal{S}^{N-1}) &= \bigoplus_{m=0}^{\infty} \mathcal{H}_m(\mathbb{R}^N) \\ W^{1,2}(\mathcal{S}^{N-1}) &= \bigoplus_{m=0}^{\infty} \mathcal{H}_m(\mathbb{R}^N) \end{aligned}$$

We are here a bit imprecise in the chosen notation. As a direct sum of vector space both direct sums are the same, but we consider them with different topologies. Furthermore to be precise the equality should be understood restricting each element of the righthand side to the sphere, \mathcal{S}^{N-1} . In the first case we equip each $\mathcal{H}_m(\mathbb{R}^N)$, with the L^2 inner product on the sphere, $\langle p, q \rangle = \int_{\mathcal{S}^{N-1}} pq$. $\mathcal{H}_m(\mathbb{R}^N)$ with this topology is a Hilbert subspace of $L^2(\mathcal{S}^{N-1})$. In the second equality we equip $\mathcal{H}_m(\mathcal{S}^{N-1})$ with the inner product of $W^{1,2}(\mathcal{S}^{N-1})$, $\langle p, q \rangle_1 = \int_{\mathcal{S}^{N-1}} pq + \int_{\mathcal{S}^{N-1}} D_\tau p \cdot D_\tau q$. With this topology $\mathcal{H}_m(\mathbb{R}^N)$ is a Hilbert subspace of $W^{1,2}(\mathcal{S}^{N-1})$.

Proof. The finite dimensional linear subspaces $\mathcal{H}_m(\mathbb{R}^N), \mathcal{H}_n(\mathbb{R}^N)$ are orthogonal with respect to both inner products $\langle \cdot, \cdot \rangle, \langle \cdot, \cdot \rangle_1$ for $m \neq n$. This is a consequence of (A.15).

Finally the restriction of polynomials to the sphere are dense in $L^2(\mathcal{S}^{N-1}) \supset W^{1,2}(\mathcal{S}^{N-1})$ due to the Stone-Weierstrass theorem. This proves the theorem since the right hand side is dense in the left. □

Combining (A.16) together with theorem A.12 shows that every $u \in L^2(\mathcal{S}^{N-1})$ has a unique decomposition $u = \sum_{m=0}^{\infty} p_m$ with $p_m \in \mathcal{H}_m(\mathbb{R}^N)$ and

$$(A.18) \quad \|u\|^2 = \sum_{m=0}^{\infty} \|p_m\|^2.$$

Furthermore u is an element of $W^{1,2}(\mathcal{S}^{N-1})$ if and only if

$$(A.19) \quad \infty > \int_{\mathcal{S}^{N-1}} |D_\tau u|^2 = \sum_{m=0}^{\infty} \int_{\mathcal{S}^{N-1}} |D_\tau p_m|^2 = \sum_{m=0}^{\infty} m^2 \left(1 + \frac{N-1}{m}\right) \|p_m\|^2.$$

This suggests an extension for defining Sobolev spaces on \mathcal{S}^{N-1} with noninteger order.

Definition A.1. For a real $s \geq 0$

$$(A.20) \quad H^s(\mathcal{S}^{N-1}) = \left\{ u = \sum_{m=0}^{\infty} p_m \in L^2(\mathcal{S}^{N-1}) : \sum_{m=0}^{\infty} m^{2s} \|p_m\|^2 < \infty \right\}.$$

Now (A.19) reads:

Corollary A.13.

$$(A.21) \quad H^1(\mathcal{S}^{N-1}) = W^{1,2}(\mathcal{S}^{N-1}).$$

As a consequence of corollary A.13 we will see that (A.20) provides an equivalent characterisation of the fractional Sobolev spaces:

Lemma A.14.

$$(A.22) \quad H^s(\mathcal{S}^{N-1}) = W^{s,2}(\mathcal{S}^{N-1}) = (W^{1,2}(\mathcal{S}^{N-1}), L^2(\mathcal{S}^{N-1}))_{1-s,2}$$

We postpone the proof after the next lemma.

Identifying interpolation spaces between $W^{1,2}(\mathcal{S}^{N-1})$ and $L^2(\mathcal{S}^{N-1})$ is now the same question as interpolating between some direct sums of Hilbert spaces with weights. This can be settled easily in a more general setting. Our presentation follows the L^2 equivalent of L. Tartar in [23, chapter 23].

We consider the situation of a direct sum of Hilbert spaces:

$$(A.23) \quad H = \bigoplus_{m=0}^{\infty} H_m$$

Lemma A.15. For a sequence of positive numbers $w = \{w_m\}_{m=0}^{\infty}$, let

$$(A.24) \quad E(w) = \left\{ a = (a_m)_m \in H : \sum_{m=0}^{\infty} w_m \|a_m\|^2 < \infty \right\} \text{ with } \|a\|_w^2 = \sum_{m=0}^{\infty} w_m \|a_m\|^2.$$

If $w(0) = \{w_m(0)\}_m$, $w(1) = \{w_m(1)\}_m$ are two such sequences, then for $0 < \theta < 1$ one has

$$(A.25) \quad (E(w(0)), E(w(1)))_{\theta,2} = E(w(\theta)) \text{ where } w_m(\theta) = w_m(0)^{1-\theta} w_m(1)^\theta.$$

Proof. We use a variant of the K -functional, namely

$$K_2(t, a) = \inf_{a=b+c} \left(\|b\|_{w(0)}^2 + t^2 \|c\|_{w(1)}^2 \right)^{\frac{1}{2}};$$

hence $K_2(t, a) \leq K(t, a) \leq \sqrt{2} K_2(t, a)$. Now for $a = \sum_m a_m$ we have $K_2(t, a)^2 = \inf_{a_m=b_m+c_m} \sum_{m=0}^{\infty} w_m(0) \|b_m\|_{w(0)}^2 + t^2 w_m(1) \|c_m\|_{w(1)}^2$. We can calculate $K_2(t, a)$ explicitly, because one is led to choose $b_m = \lambda_m a_m + d_m$ with $d_m \in H_m \cap \text{span}(a_m)^\perp$. Then $c_m = (1 - \lambda_m) a_m - d_m$ and so $\|b_m\|^2 = \lambda_m^2 \|a_m\|^2 + \|d_m\|^2$, $\|c_m\|^2 = (1 -$

$\lambda_m)^2 \|a_m\|^2 + \|d_m\|$. Hence $d_m = 0$ and one is led to choose for b_m the value λ_m that minimises $w_m(0)\lambda_m^2 \|a_m\|^2 + t^2 w_m(1)(1 - \lambda_m)^2 \|a_m\|^2$. One finds

$$\lambda_m = \frac{t^2 w_m(1)}{w_m(0) + t^2 w_m(1)} \text{ and } 1 - \lambda_m = \frac{w_m(0)}{w_m(0) + t^2 w_m(1)};$$

so $K_2(t, a)$ is computed explicitly by

$$K_2(t, a)^2 = \sum_{m=0}^{\infty} \|a_m\|^2 t^2 \frac{w_m(0)w_m(1)}{w_m(0) + t^2 w_m(1)}.$$

Finally Lebesgue's monotone convergence theorem provides

$$\|t^{-\theta} K_2(t, a)\|_{L^2(\mathbb{R}_+, \frac{dt}{t})}^2 = \sum_{m=0}^{\infty} \|a_m\|^2 \int_0^{\infty} t^{2(1-\theta)} \frac{w_m(0)w_m(1)}{w_m(0) + t^2 w_m(1)} \frac{dt}{t},$$

making the change of variables $t = \sqrt{\frac{w_m(0)}{w_m(1)}} s$, one finds

$$\int_0^{\infty} t^{2(1-\theta)} \frac{w_m(0)w_m(1)}{w_m(0) + t^2 w_m(1)} \frac{dt}{t} = w_m(0)^{1-\theta} w_m(1)^{\theta} \int_0^{\infty} \frac{s^{1-2\theta}}{1 + s^2} ds.$$

Since $C = \int_0^{\infty} \frac{s^{1-2\theta}}{1 + s^2} ds = \frac{\pi}{2 \sin(\pi\theta)}$, this gives

$$\|t^{-\theta} K_2(t, a)\|_{L^2(\mathbb{R}_+, \frac{dt}{t})}^2 = C \sum_{m=0}^{\infty} w_m(\theta) \|a_m\|^2.$$

□

Proof of lemma A.14. There is unique decomposition $L^2(\mathcal{S}^{N-1}) \rightarrow \bigoplus_m \mathcal{H}_m(\mathbb{R}^N)$ with $u \mapsto \{p_m\}_m$ and $u = \sum_m p_m$ as seen in theorem A.12. This map is an isometry between $L^2(\mathcal{S}^{N-1})$ and $H^0(\mathcal{S}^{N-1})$ and continuously linear between $W^{1,2}(\mathcal{S}^{N-1})$ and $H^1(\mathcal{S}^{N-1})$. Thus lemma A.15 showed that the decomposition is a linear homeomorphism between

$$W^{s,2}(\mathcal{S}^{N-1}) = (W^{1,2}(\mathcal{S}^{N-1}), L^2(\mathcal{S}^{N-1}))_{1-s,2}$$

and

$$(H^1(\mathcal{S}^{N-1}), H^0(\mathcal{S}^{N-1}))_{1-s,2} = H^s(\mathcal{S}^{N-1});$$

that is the statement of lemma A.14. □

Now we come to the second part estimating the energy of the solution to the Dirichlet problem on $A_{1,R} = B_1 \setminus B_R$ for a fixed $0 < R < 1$. We start with estimating them for polynomials and after that we will use these estimates to conclude it for general functions.

Consider the following Dirichlet problem:

Let $p, q \in \mathcal{H}_m(\mathbb{R}^N)$ be given, and let $P : A_{1,R} \rightarrow \mathbb{R}$ be the unique solution of

$$(A.26) \quad \begin{cases} \Delta P = 0, & \text{on } A_{1,R} \\ P(x) = p(x) \text{ and } P(Rx) = q(x) & \text{for all } x \in \mathcal{S}^{N-1} \end{cases}$$

Lemma A.16. *Let p, q be two given constants, i.e. $p, q \in \mathcal{H}_0(\mathbb{R}^N)$, then there are $\tilde{p}, \tilde{q} \in \mathcal{H}_0(\mathbb{R}^N)$ s.t. the solution P of (A.26) is*

$$(A.27) \quad P(x) = \begin{cases} \tilde{p} + \tilde{q} \ln(r), & \text{if } N = 2 \\ \tilde{p} + \frac{\tilde{q}}{|x|^{N-2}}, & \text{if } N > 2; \end{cases}$$

furthermore we have the estimate

$$(A.28) \quad \int_{A_{1,R}} |DP|^2 = \begin{cases} \frac{2\pi}{-\ln(R)} |p - q|^2, & \text{if } N = 2 \\ \frac{N(N-2)\omega_N}{R^{2-N}-1} |p - q|^2, & \text{if } N > 2; \end{cases}$$

Proof. It is a standard calculation that $\ln(r)$ for $N = 2$ and $|x|^{2-N}$ for $N > 2$ are harmonic on $\mathbb{R}^N \setminus \{0\}$, hence the $P(x) = P(r)$ defined by (A.27) are harmonic. The boundary conditions in (A.26) translate to

$$P(1) = p \text{ hence } \tilde{p} = p \text{ for } N = 2 \text{ and } \tilde{p} + \tilde{q} = p \text{ for } N > 2$$

$$P(R) = q \text{ hence } \tilde{p} + \tilde{q} \ln(R) = q \text{ for } N = 2 \text{ and } \tilde{p} + \frac{\tilde{q}}{R^{2-N}} = q \text{ for } N > 2.$$

In the case of $N = 2$ one solves for $\tilde{q} = \frac{q-p}{\ln(R)}$, in the case of $N > 3$ for $\tilde{q} = \frac{q-p}{R^{2-N}-1}$. Apply Green's formula on the annulus and then insert the boundary conditions in the second to obtain:

$$(A.29) \quad \int_{A_{1,R}} |DP|^2 = \int_{\partial A_{1,R}} P \frac{\partial P}{\partial \nu} = \int_{S^{N-1}} P(x) \frac{\partial P}{\partial r}(x) - \int_{\partial B_R} P(x) \frac{\partial P}{\partial r}(x) \\ = \int_{S^{N-1}} p(x) \frac{\partial P}{\partial r}(x) - \int_{\partial B_R} q(R^{-1}x) \frac{\partial P}{\partial r}(x).$$

For $N = 2$, $\frac{\partial P}{\partial r}(r) = \frac{\tilde{q}}{r}$ otherwise $\frac{\partial P}{\partial r}(r) = \frac{(2-N)\tilde{q}}{r^{N-1}}$, hence in two dimensions we found

$$\int_{\partial A_{1,R}} P \frac{\partial P}{\partial \nu} = 2\pi \left(p \frac{\partial P}{\partial r}(1) - q \frac{\partial P}{\partial r}(R)R \right) = \frac{2\pi}{-\ln R} |p - q|^2;$$

in higher dimensions

$$\int_{\partial A_{1,R}} P \frac{\partial P}{\partial \nu} = N\omega_N \left(p \frac{\partial P}{\partial r}(1) - q \frac{\partial P}{\partial r}(R)R^{N-1} \right) = \frac{N(N-2)\omega_N}{R^{2-N}-1} |p - q|^2.$$

□

For the estimates in the case $m \geq 1$ we introduce two functions:

$$(A.30) \quad f(t) = \frac{\cosh((N-1)t) - 1}{\sinh(t)} \\ \tilde{f}(t) = \frac{t}{t - t^{-1}}.$$

Lemma A.17. *Let $p, q \in \mathcal{H}_m(\mathbb{R}^N)$, $m > 0$, be given. Then there are $\tilde{p}, \tilde{q} \in \mathcal{H}_m(\mathbb{R}^N)$ s.t. that the solution to (A.26) has the form*

$$(A.31) \quad P(x) = \tilde{p}(x) + K[\tilde{q}](x) = \tilde{p}(x) + \frac{\tilde{q}(x)}{|x|^{N+2m-2}};$$

furthermore we can estimate the energy either by

$$(A.32) \quad \int_{A_{1,R}} |DP|^2 - \frac{2m+N-2}{R^{-m-N+2}-R^m} \|p - q\|^2 \leq f(\ln(R^{-m})) m \left(\|p\|^2 + \|q\|^2 \right);$$

or by

$$(A.33) \quad \int_{A_{1,R}} |DP|^2 \leq 4N\tilde{f}(R^{-m}) m \left(\|p\|^2 + \|q\|^2 \right).$$

Proof. The Kelvin transform maps harmonic polynomials $\tilde{q} \in \mathcal{H}_m(\mathbb{R}^N)$ to harmonic functions on $\mathbb{R}^N \setminus \{0\}$, homogeneous of degree $2-N-m$. Hence P defined by (A.31) is harmonic on $\mathbb{R}^N \setminus \{0\}$. The boundary conditions impose $\tilde{p}(x) + \tilde{q}(x) = p(x)$ and $R^m \tilde{p}(x) + R^{2-N-m} \tilde{q}(x) = q(x)$. Solving this for \tilde{p} and \tilde{q} gives

$$\tilde{p}(x) = \frac{R^{2-N-m}p(x) - q(x)}{R^{2-N-m} - R^m} \text{ and } \tilde{q}(x) = \frac{q(x) - R^m p(x)}{R^{2-N-m} - R^m}.$$

As before we can use the Euler formula for homogenous function u of degree λ , $r \frac{\partial u(x)}{\partial r} = \lambda u(x)$, to simplify the integrals and inserting $P(x) = p(x)$, $P(Rx) = q(x)$ for all $x \in \mathcal{S}^{N-1}$ we obtain

$$\begin{aligned} \int_{\partial A_{1,R}} P \frac{\partial P}{\partial \nu} &= \int_{\mathcal{S}^{N-1}} p(x) DP(x) \cdot x - R^{N-2} \int_{\mathcal{S}^{N-1}} q(x) DP(Rx) \cdot Rx \\ &= \int_{\mathcal{S}^{N-1}} p(x) (m\tilde{p}(x) + (2 - N - m)\tilde{q}(x)) \\ &\quad - R^{N-2} \int_{\mathcal{S}^{N-1}} q(x) (mR^m\tilde{p}(x) + (2 - N - m)R^{2-N-m}\tilde{q}(x)) \\ &= \frac{m}{R^{2-N-m} - R^m} \left(\left[R^{2-N-m} + \left(1 + \frac{N-2}{m}\right) R^m \right] \|p\|^2 \right. \\ &\quad \left. + \left[R^{m+N-2} + \left(1 + \frac{N-2}{m}\right) R^{-m} \right] \|q\|^2 - \left(2 + \frac{N-2}{m}\right) 2\langle p, q \rangle \right). \end{aligned}$$

To obtain the first estimate (A.32), subtract $\frac{2m+N-2}{R^{-m-N+2}-R^m} \|p-q\|^2$ from the integral above and use $-2\langle p, q \rangle = \|p-q\|^2 - \|p\|^2 - \|q\|^2$, which gives

$$\begin{aligned} -\left(2 + \frac{N-2}{m}\right) 2\langle p, q \rangle &= \frac{1}{m} (2m + N - 2) \|p - q\|^2 \\ &\quad - \|p\|^2 - \left(1 + \frac{N-2}{m}\right) \|p\|^2 - \|q\|^2 - \left(1 + \frac{N-2}{m}\right) \|q\|^2. \end{aligned}$$

We then conclude

$$\begin{aligned} \int_{A_{1,R}} |DP|^2 - \frac{2m+N-2}{R^{-m-N+2}-R^m} \|p-q\|^2 &= \\ \frac{m}{R^{-m-N+2}-R^m} \left((R^{-m-N+2} - 1) + \left(1 + \frac{N-2}{m}\right) (R^m - 1) \right) \|p\|^2 \\ + \frac{m}{R^{-m-N+2}-R^m} \left((R^{m+N-2} - 1) + \left(1 + \frac{N-2}{m}\right) (R^{-m} - 1) \right) \|q\|^2. \end{aligned}$$

One easily checks that the function $g(y) = (y^a - 1) - a(y - 1)$ (defined for $y > 0$ and $a > 1$) attains its minimum at $y = 1$: $g(1) = 0$ i.e. $a(y - 1) \leq y^a - 1$. In our case that gives $(1 + \frac{N-2}{m})(R^m - 1) \leq (R^{m+N-2} - 1)$ and $(1 + \frac{N-2}{m})(R^{-m} - 1) \leq (R^{-m-N+2} - 1)$. Hence we can simplify to

$$\begin{aligned} \int_{A_{1,R}} |DP|^2 - \frac{2m+N-2}{R^{-m-N+2}-R^m} \|p-q\|^2 \\ \leq m \frac{R^{2-N-m} + R^{m+N-2} - 2}{R^{2-N-m} - R^m} (\|p\|^2 + \|q\|^2) \leq m f(\ln(R^{-m})) (\|p\|^2 + \|q\|^2); \end{aligned}$$

where we used

$$\frac{R^{2-N-m} + R^{m+N-2} - 2}{R^{2-N-m} - R^m} \leq \frac{\cosh\left(\left(1 + \frac{N-2}{m}\right) \ln(R^{-m})\right) - 1}{\sinh(\ln(R^{-m}))} \leq f(\ln(R^{-m}))$$

with $R^{2-N-m} - R^m \geq R^{-m} - R^m$.

To deduce (A.33), we estimate quite brutally $-2\langle p, q \rangle \leq \|p\|^2 + \|q\|^2$. As coefficient in front of $\|p\|^2$ we get

$$\begin{aligned} \frac{R^{2-N-m} + \left(1 + \frac{N-2}{m}\right) R^m + \left(2 + \frac{N-2}{m}\right)}{R^{2-N-m} - R^m} \\ \leq \frac{2\left(2 + \frac{N-2}{m}\right) R^{-m}}{R^{-m} - R^{m+N-2}} \leq 4N \frac{R^{-m}}{R^{-m} - R^m}. \end{aligned}$$

In the last inequality we used that $R^{-m} - R^{m+N-2} \geq \frac{1}{2}(R^{-m} - R^m)$. This can be checked as follows: $y \in]0, 1] \mapsto (y^{-1} - y^a) - \frac{1}{2}(y^{-1} - y)$ for $a \geq 1$ is nonincreasing and vanishes for $y = 1$; the inequality follows inserting $y = R^m$ and $a = 1 + \frac{N-2}{m}$. The coefficient in front of $\|q\|^2$ is

$$\begin{aligned} & \frac{R^{m+N-2} + \left(1 + \frac{N-2}{m}\right) R^{-m} + \left(2 + \frac{N-2}{m}\right)}{R^{2-N-m} - R^m} \\ & \leq \frac{2 \left(2 + \frac{N-2}{m}\right) R^{-m}}{R^{2-N-m} - R^m} \leq 4N \frac{R^{-m}}{R^{-m} - R^m}. \end{aligned}$$

This completes the proof. \square

To conclude the interpolation theorem we need shortly to analyse the behaviour of the two functions f and \tilde{f} in (A.30).

Lemma A.18. *f is monotone increasing, hence $f(\ln(R^{-m}))$ is increasing in m and decreasing in $R \in]0, 1]$. Furthermore we have $\lim_{y \searrow 0} f(y) = 0$; \tilde{f} is monotone decreasing, hence for $\delta > 0$, $m^{-2\delta} \tilde{f}(R^{-m})$ is decreasing in m and increasing in $R \in]0, 1]$. Furthermore we have $m^{-2\delta} \tilde{f}(e^{m^{-\delta}}) \leq \frac{2}{m^\delta} \rightarrow 0$ as $m \rightarrow \infty$.*

Proof. f' is given by

$$f'(y) = \frac{g(N-1, y)}{\sinh^2(y)};$$

where we introduced the function

$$g(a, y) = a \sinh(ay) \sinh(y) - \cosh(y)(\cosh(ay) - 1) \text{ for } a \geq 1, y > 0$$

f' is strictly positive because firstly we have $g(1, y) = \sinh^2(y) - \cosh^2(y) + \cosh(y) = \cosh(y) - 1 > 0$ for $y > 1$ and secondly

$$\begin{aligned} \frac{\partial g}{\partial a}(a, y) &= \sinh(ay) \sinh(y) + ay \cosh(ay) \sinh(y) - y \cosh(y) \sinh(ay) \\ &\geq \sinh(ay) \sinh(y) + y(\cosh(ay) \sinh(y) - \cosh(y) \sinh(ay)) \\ &= \sinh(ay) \sinh(y) - y \sinh((a-1)y) \\ &\geq y(\sinh(ay) - \sinh((a-1)y)) \geq 0. \end{aligned}$$

We used the addition theorem and $\sinh(y) \geq y$ for $y \geq 0$. Therefore we found $g((N-1), y) \geq g(1, y) > 0$. Using L'Hospital's rule we have

$$\lim_{y \searrow 0} f(y) = \frac{(N-1) \sinh((N-1)0)}{\cosh(0)} = 0.$$

$\tilde{f}'(y) = \frac{-2y^{-1}}{(y-y^{-1})^2} < 0$, hence \tilde{f} is monotone decreasing and so is $m \mapsto m^{-2\delta}$. Finally the conclusions on the behaviour of $f(\ln(R^{-m}))$ and $m^{-2\delta} \tilde{f}(R^{-m})$ follow because for $0 < R \leq 1$ we have $m \mapsto \ln(R^{-m})$ is monotone increasing and $R \in]0, 1] \mapsto \ln(R^{-m})$ monotone decreasing. The last estimate just follows from $\sinh(y) \geq y$:

$$m^{-2\delta} \tilde{f}(e^{m^{-\delta}}) = \frac{e^{m^{-\delta}}}{2m^{2\delta} \sinh(m^{-\delta})} \leq \frac{2}{m^\delta}.$$

\square

Now we are able to prove the interpolation lemma A.9:

Proof of Lemma A.9. Recall that $\epsilon > 0$ and $1 > s > \frac{1}{2}$ are given. Fix $\delta = s - \frac{1}{2} > 0$. Lemma A.14 stated that $W^{s,2}(\mathcal{S}^{N-1}) = H^s(\mathcal{S}^{N-1})$ and each element of $H^s(\mathcal{S}^{N-1})$, a subset of the vector space $\bigoplus_{m=0}^{\infty} \mathcal{H}_m(\mathbb{R}^N)$. Therefore it is sufficient to proof (A.13) under the additional assumption that for some finite large M we have $u =$

$\sum_{m=0}^M p_m, v = \sum_{m=0}^M q_m$ for $p_m, q_m \in \mathcal{H}_m(\mathbb{R}^N)$. But we have to ensure that the constant in (A.13) is independent of M .

Firstly observe, that if P_m, P_n are the solutions to (A.26) corresponding to pairs $p_m, q_m \in \mathcal{H}_m(\mathbb{R}^N)$, $p_n, q_n \in \mathcal{H}_n(\mathbb{R}^N)$ constructed in the preparatory lemmas A.16, A.17. Hence we deduce (as in the proofs to lemma A.16, A.17, using the Euler formula)

$$\begin{aligned} \int_{\partial A_{1,R}} P_n \frac{\partial P_m}{\partial \nu} &= \int_{S^{N-1}} p_n(x) DP_m(x) \cdot x - R^{N-2} \int_{S^{N-1}} q_n(x) DP_m(Rx) \cdot Rx \\ &= m \langle p_n, \tilde{p}_m \rangle + (2 - N - m) \langle p_n, \tilde{q}_m \rangle \\ &\quad - R^{N-2} (m R^m \langle q_n, \tilde{p}_m \rangle + (2 - N - m) R^{2-N-m} \langle q_n, \tilde{q}_m \rangle) \\ &= 0 \end{aligned}$$

due to the orthogonality (A.15). To every $0 \leq m \leq M$ let P_m be the solution of (A.26) to the pair $p_m, q_m \in \mathcal{H}_m(\mathbb{R}^N)$ given by the decompositions $u = \sum_{m=0}^M p_m, v = \sum_{m=0}^M q_m$. For $P = \sum_{m=0}^M P_m$ we have just shown that

$$\int_{A_{1,R}} |DP|^2 = \int_{\partial A_{1,R}} P \frac{\partial P}{\partial \nu} = \sum_{m=0}^M \int_{\partial A_{1,R}} P_m \frac{\partial P_m}{\partial \nu}.$$

Let us define $R_\epsilon = e^{-m_\epsilon^{-1-\delta}}$ for some sufficiently large $m_\epsilon > 1$ with the property that $f(y) < \epsilon$ for $0 < y < (m_\epsilon - 1)^{-\delta}$ and $4N \frac{2}{m_\epsilon^\delta} < \epsilon$. Such an m_ϵ exists as a consequence of lemma A.18.

Finally for any $R_\epsilon \leq R < 1$ we may fix $m_R \geq m_\epsilon$ s.t. $e^{-(m_R-1)^{-1-\delta}} < R \leq e^{-m_R^{-1-\delta}}$. Using the results of lemma A.18 we conclude for $m \geq m_R$

$$m^{-2\delta} \tilde{f}(R^{-m}) \leq m_R^{-2\delta} \tilde{f}(R^{-m_R}) \leq m_R^{-2\delta} \tilde{f}((e^{-m_R^{-1-\delta}})^{-m_R}) < \frac{\epsilon}{4N}.$$

And for $m < m_R$ i.e. $m \leq m_R - 1$ we deduce

$$\begin{aligned} f(\ln(R^{-m})) &\leq f(\ln(R^{-(m_R-1)})) \\ &\leq f(-(m_R - 1) \ln(e^{-(m_R-1)^{-1-\delta}})) = f((m_R - 1)^{-\delta}) < \epsilon. \end{aligned}$$

Finally we fix the constant $C = C(\epsilon, R)$ to be the maximum of the constants of lemma A.16 i.e. $\frac{2\pi}{\ln(R)}$ for $N = 2$, $\frac{N(N-2)\omega_N}{R^{2-N-1}}$ for $N > 2$ and the one of (A.32) i.e.

$$\frac{2m+N-2}{R^{-m-N+2}-R^m} \text{ for } m \leq m_R.$$

We have shown that

$$\int_{\partial A_{1,R}} P_m \frac{\partial P_m}{\partial \nu} \leq \epsilon m^{2s} (\|p_m\|^2 + \|q_m\|^2) \text{ for } m \geq m_R;$$

and

$$\int_{\partial A_{1,R}} P_m \frac{\partial P_m}{\partial \nu} \leq \epsilon m (\|p_m\|^2 + \|q_m\|^2) + C \|p_m - q_m\|^2 \text{ for } m < m_R.$$

This proves a first version of the interpolation since we found

$$\begin{aligned} \int_{A_{1,R}} |DP|^2 &= \sum_{m=0}^{m_R-1} \int_{\partial A_{1,R}} P_m \frac{\partial P_m}{\partial \nu} + \sum_{m=m_R}^M \int_{\partial A_{1,R}} P_m \frac{\partial P_m}{\partial \nu} \\ &\leq \epsilon \sum_{m=0}^M m^{2s} (\|p_m\|^2 + \|q_m\|^2) + C \sum_{m=0}^{m_R-1} \|p_m - q_m\|^2 \\ &\leq \epsilon \sum_{m=0}^{\infty} m^{2s} (\|p_m\|^2 + \|q_m\|^2) + C \sum_{m=0}^{\infty} \|p_m - q_m\|^2; \end{aligned}$$

the right hand side is independent of M , so that we can pass to the limit as $M \rightarrow \infty$. Although $\sum_{m=1}^{\infty} m^{2s} \|p_m\|^2$ does not contain the 0th. order lemma, A.14 provides only equivalence for complete norms. Choosing $\epsilon > 0$ (a priori smaller, if necessary, to absorb the constants) we got, for any admissible $W^{s,2}$ -norm:

$$\int_{A_{1,R}} |Dw|^2 \leq \epsilon \left(\|u\|_{W^{s,2}(\mathcal{S}^{N-1})}^2 + \|v\|_{W^{s,2}(\mathcal{S}^{N-1})}^2 \right) + C \|u - v\|_{\mathcal{S}^{N-1}}^2.$$

To pass actually to (A.13) we can use a small observation and the Poincaré inequality (A.8). Let $u, v \in W^{s,2}(\mathcal{S}^{N-1})$ be given, apply the so far obtained interpolation to $\tilde{u} = u - \frac{1}{2}(f_{\mathcal{S}^{N-1}} u + f_{\mathcal{S}^{N-1}} v)$ and $\tilde{v} = v - \frac{1}{2}(f_{\mathcal{S}^{N-1}} u + f_{\mathcal{S}^{N-1}} v)$ providing $\tilde{w} \in W^{1,2}(A_{1,R})$. $\tilde{w} = w + \frac{1}{2}(f_{\mathcal{S}^{N-1}} u + f_{\mathcal{S}^{N-1}} v)$ has the desired properties because

$$\begin{aligned} \|\tilde{u}\|_{W^{s,2}(\mathcal{S}^{N-1})}^2 &= \|\tilde{u}\|_{L^2(\mathcal{S}^{N-1})}^2 + \|\tilde{u}\|_{s,\mathcal{S}^{N-1}}^2 \\ &= \|\tilde{u}\|_{L^2(\mathcal{S}^{N-1})}^2 + \|u\|_{s,\mathcal{S}^{N-1}}^2 \end{aligned}$$

and by the Poincaré inequality (A.8) and $2\tilde{u} = (u - f_{\mathcal{S}^{N-1}} u) + (v - f_{\mathcal{S}^{N-1}} v) + (u - v)$

$$2\|\tilde{u}\|_{L^2(\mathcal{S}^{N-1})} \leq C (\|u\|_{s,\mathcal{S}^{N-1}} + \|v\|_{s,\mathcal{S}^{N-1}}) + \|u - v\|_{L^2(\mathcal{S}^{N-1})}.$$

We argue similarly for \tilde{v} . In conclusion we obtained

$$\begin{aligned} \int_{A_{1,R}} |Dw|^2 &= \int_{A_{1,R}} |D\tilde{w}|^2 \leq \epsilon \left(\|\tilde{u}\|_{W^{s,2}(\mathcal{S}^{N-1})}^2 + \|\tilde{v}\|_{W^{s,2}(\mathcal{S}^{N-1})}^2 \right) + C \|\tilde{u} - \tilde{v}\|_{\mathcal{S}^{N-1}}^2 \\ &\leq C \epsilon \left(\|u\|_{s,\mathcal{S}^{N-1}}^2 + \|v\|_{s,\mathcal{S}^{N-1}}^2 \right) + C \|u - v\|_{\mathcal{S}^{N-1}}^2. \end{aligned}$$

□

APPENDIX B. Q -VALUED FUNCTIONS

B.1. Fractional Sobolev spaces for Q -valued functions. As before we restrict ourself to $0 < s \leq 1$. Since $\mathcal{A}_Q(\mathbb{R}^n)$ fails to be a linear space, $L^2(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ is not a Banach space. Hence we are not in a setting for classical interpolation methods. Nonetheless there are two ways to define $W^{s,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ in a natural way:

- (a) using Almgren's bilipschitz embedding $\xi : \mathcal{A}_Q(\mathbb{R}^n) \rightarrow \mathbb{R}^m$, theorem 0.2.1,

$$W^{s,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n)) = \{u \in L^2(\Omega, \mathcal{A}_Q(\mathbb{R}^n)) : \xi \circ u \in W^{s,2}(\Omega, \mathbb{R}^m)\};$$

- (b) using the Gagliardo norm

$$W^{s,2}(\Omega) = \{u \in L^2(\Omega, \mathcal{A}_Q(\mathbb{R}^n)) : \|u\|_{s,\Omega}^2 = \int_{\Omega \times \Omega} \frac{\mathcal{G}(u(x), u(y))^2}{|x - y|^{N+2s}} dx dy < \infty\}.$$

The equivalence of both definitions follows from the bilipschitz property of ξ i.e $c|\xi \circ u(x) - \xi \circ u(y)| \leq \mathcal{G}(u(x), u(y)) \leq |\xi \circ u(x) - \xi \circ u(y)|$ for some $c = c(n, Q)$. This implies

$$(B.1) \quad c\|\xi \circ u\|_{s,\Omega}^2 \leq \|u\|_{s,\Omega}^2 \leq \|\xi \circ u\|_{s,\Omega}^2.$$

We had seen that all definitions of $W^{s,2}(\Omega, \mathbb{R}^m)$ are equivalent in case of a Lipschitz regular domain $\Omega \subset \mathbb{R}^N$.

Combining the definition of $W^{s,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ as suggested in (a) with (B.1) we obtain nearly all statements for single valued functions as well for multiple valued functions. For the sake of completeness we state them now for Q -valued functions:

Corollary B.1. *To any given $-1 < a < 1$ and $\frac{1}{2} < s \leq 1$ there is a constant $C >$ with the property, that if $u \in W^{s,2}(\mathcal{S}^{N-1} \cap \{x_N > a\}, \mathcal{A}_Q(\mathbb{R}^n)), v \in W^{s,2}(\mathcal{S}^{N-1} \cap \{x_N < a\}, \mathcal{A}_Q(\mathbb{R}^n))$ with $u|_{\mathcal{S}^{N-1} \cap \{x_N=a\}} = v|_{\mathcal{S}^{N-1} \cap \{x_N=a\}}$ then*

$$(B.2) \quad U(x) = \begin{cases} u(x), & \text{if } x \in \mathcal{S}^{N-1}, x_N > a \\ v(x), & \text{if } x \in \mathcal{S}^{N-1}, x_N < a \end{cases}$$

defines an element in $W^{s,2}(\mathcal{S}^{N-1}, \mathcal{A}_Q(\mathbb{R}^n))$ satisfying

$$(B.3) \quad \|U\|_{s, \mathcal{S}^{N-1}} \leq C (\|u\|_{s, \mathcal{S}^{N-1} \cap \{x_N > a\}} + \|v\|_{s, \mathcal{S}^{N-1} \cap \{x_N < a\}})$$

Lemma B.2. *Let $\frac{1}{2} < s \leq 1$ and $\epsilon > 0$ be given then there exists $R_\epsilon > 0$ with the property: for any $R_\epsilon \leq R < 1$ there is $C = C(\epsilon, R, n, Q)$ s.t. given $u, v \in W^{s,2}(\mathcal{S}^{N-1}, \mathcal{A}_Q(\mathbb{R}^n))$ one can find $w \in W^{1,2}(A_{1,R}, \mathcal{A}_Q(\mathbb{R}^n))$ on the annulus $A_{1,R} = B_1 \setminus B_R$ with $w(x) = u(x)$ and $w(Rx) = v(x)$ for $x \in \mathcal{S}^{N-1}$ that satisfies*

$$(B.4) \quad \int_{A_{1,R}} |Dw|^2 \leq \epsilon (\|u\|_{s, \mathcal{S}^{N-1}}^2 + \|v\|_{s, \mathcal{S}^{N-1}}^2) + C \|\mathcal{G}(u, v)\|_{\mathcal{S}^{N-1}}^2.$$

Proof. For $s = 1$ we set $R_\epsilon = 1 - \epsilon$. We obtain $\tilde{w} \in W^{1,2}(A_{1,R}, \mathbb{R}^m)$ applying observation (A.12) to $\xi \circ u, \xi \circ v$ with $\epsilon' = 1 - R, R_\epsilon < R < 1$. We obtain $\tilde{w} \in W^{1,2}(A_{1,R}, \mathbb{R}^m)$. The retraction $w = \rho \circ \tilde{w} \in W^{1,2}(A_{1,R}, \mathcal{A}_Q(\mathbb{R}^n))$ then has up to a constant the desired properties.

For $\frac{1}{2} < s < 1$ we proceed similarly. Firstly apply lemma A.9 to $\xi \circ u, \xi \circ v$ that gives $\tilde{w} \in W^{1,2}(A_{1,R}, \mathbb{R}^m)$. As before the retraction $w = \rho \circ \tilde{w} \in W^{1,2}(A_{1,R}, \mathcal{A}_Q(\mathbb{R}^n))$ fulfils up to a constant the desired properties. \square

B.2. Dirichlet minimizers on cylinders, Remark 1.5.1. As announced in Remark 1.5.1 we present the proof given in [12] to the following observation.

Lemma B.3. *$u(x) \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ and $U(x, t) = u(x)$ is Dirichlet minimizing on $\Omega \times \mathbb{R}$ then u itself is minimizing in Ω*

Proof. Given an arbitrary competitor $v(x) \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ to u i.e. $u|_{\partial\Omega} = v|_{\partial\Omega}$ on $\partial\Omega$. We fix an interpolation $w \in W^{1,2}(\Omega \times [0, 1], \mathcal{A}_Q(\mathbb{R}^n))$ satisfying $w(x, 0) = u(x), w(x, 1) = v(x)$ for all $x \in \Omega$ and $w(x, t) = u|_{\partial\Omega}(x) = v|_{\partial\Omega}(x)$ on $\partial\Omega \times [0, 1]$.

$$V(x, t) = \begin{cases} w(x, L+1-t) & \text{if } L \leq t \leq L+1 \\ v(x) & \text{if } -L \leq t \leq L \\ w(x, L+1+t) & \text{if } -L-1 \leq t \leq -L. \end{cases}$$

defines an admissible competitor to U . Hence the minimality of U ensures

$$\begin{aligned} 2(L+1) \int_{\Omega} |Du|^2 &= \int_{\Omega \times [-L-1, L+1]} |DU|^2 \\ &\leq \int_{\Omega \times [-L-1, L+1]} |DV|^2 = 2L \int_{\Omega} |Dv|^2 + 2 \int_{\Omega \times [0, 1]} |Dw|^2. \end{aligned}$$

This is equivalent to

$$\int_{\Omega} |Du|^2 \leq \left(1 - \frac{1}{L+1}\right) \int_{\Omega} |Dv|^2 + \frac{1}{L+1} \int_{\Omega \times [0, 1]} |Dw|^2.$$

for all $L \geq 0$, proving the minimality of u . \square

B.3. $W^{s,p}$ -selection for $s > \frac{1}{p}$. The proof of this lemma is due to Camillo De Lellis, but has not been published so far.

Lemma B.4. *Let $s > \frac{1}{p}$, $Q \in \mathbb{N}$ be given, then for $u \in W^{s,p}([0,1], \mathcal{A}_Q(\mathbb{R}^n))$ we can find $v = (v_1, \dots, v_Q) : [0,1] \rightarrow (\mathbb{R}^n)^Q$ with the property that*

(i)

$$[v(t)] = \sum_{i=1}^Q \llbracket v_i(t) \rrbracket = u(t) \text{ for all } t \in [0,1];$$

(ii) $v \in W^{s',p}([0,1], (\mathbb{R}^n)^Q)$ for any $s' < s$ i.e. there is a positive constant C depending on Q and p, s, s' s.t.

$$\int_{[0,1] \times [0,1]} \frac{|v(x) - v(y)|^p}{|x - y|^{1+ps'}} dx dy \leq C \int_{[0,1] \times [0,1]} \frac{\mathcal{G}(u(x), u(y))^p}{|x - y|^{1+ps}} dx dy$$

Proof. The lemma is a consequence of the results on regular selections of multivalued functions, [11, theorem 1.1], and the following estimate

$$(B.5) \quad \int_{0 \leq x \leq y \leq 1} \frac{\max_{\sigma, \tau \in [x,y]} |f(\sigma) - f(\tau)|^p}{|x - y|^{1+ps'}} dx dy \leq C \int_{0 \leq \sigma \leq \tau \leq 1} \frac{|f(\sigma) - f(\tau)|^p}{|\sigma - \tau|^{1+ps}} d\sigma d\tau$$

for a constant C depending only on $p, s' < s$.

We start with proving (B.5). $W^{s,p}([0,1]) \subset C^{0, s - \frac{1}{p}}([0,1])$ for $ps > 1$ i.e. for any $\sigma, \tau \in [0,1]$

$$(B.6) \quad |f(\sigma) - f(\tau)| \leq C \llbracket f \rrbracket_{s,p,[0,1]}$$

where we used the abbreviation $\llbracket f \rrbracket_{s,p,[a,b]}^p = \int_{[a,b] \times [a,b]} \frac{|f(x) - f(y)|^p}{|x - y|^{1+ps}} dx dy$. This holds by standard theory. Or it may be concluded from lemma 1.4.2. To do so extend f to $\tilde{f} \in W^{s,p}([-1,3], \mathbb{R}^n)$ by

$$\tilde{f} = \begin{cases} f(-t), & \text{if } -1 < t < 0 \\ f(t), & \text{if } 0 < t < 1 \\ f(1-t), & \text{if } 1 < t < 2. \end{cases}$$

The means $\tilde{f}(x, r) = f_{x-r}^{x+r} \tilde{f}$ are well-defined for all $x \in [0,1]$ and $r < 1$. (B.6) for \tilde{f} in the case of $p = 2$ agrees with (1.4.3) in lemma 1.4.2 since (1.4.2) is satisfied with $\beta = \frac{1}{2}$; for general p the calculations have to be adapted classically. We conclude: for all $\sigma, \tau \in [0,1]$

$$|f(\sigma) - f(\tau)| = |\tilde{f}(\sigma) - \tilde{f}(\tau)| \leq C \llbracket \tilde{f} \rrbracket_{s,p,[-1,2]} \leq C \llbracket f \rrbracket_{s,p,[0,1]}.$$

For any $f \in W^{s,p}([a,b], \mathbb{R}^n)$ we may applying (B.6) to the rescaled function $f_{a,\rho}(t) = f(a + \rho t)$ with $\rho = b - a$:

$$\begin{aligned} \max_{x,y \in [a,b]} |f(x) - f(y)| &= \max_{\sigma, \tau \in [0,1]} |f_{a,\rho}(\sigma) - f_{a,\rho}(\tau)| \leq C \llbracket f_{a,\rho} \rrbracket_{s,p,[0,1]} \\ &= C \rho^{s - \frac{1}{p}} \llbracket f \rrbracket_{s,p,[a,b]} = C(b-a)^{s - \frac{1}{p}} \llbracket f \rrbracket_{s,p,[a,b]}. \end{aligned}$$

Inserting this in the left hand side of (B.5) gives

$$\begin{aligned}
& \int_{0 \leq x \leq y \leq 1} \frac{\max_{\sigma, \tau \in [x, y]} |f(\sigma) - f(\tau)|^p}{|x - y|^{1+ps'}} dx dy \\
& \leq C \int_{0 \leq x \leq y \leq 1} \frac{(y-x)^{ps-1}}{(y-x)^{1+ps}} \int_{x \leq \sigma \leq \tau \leq 1} \frac{|f(\sigma) - f(\tau)|^p}{(\tau - \sigma)^{1+ps}} d\tau d\sigma dx dy \\
& \leq C \int_{0 \leq \sigma \leq \tau \leq 1} \left(\int_0^\sigma \int_\tau^1 (y-x)^{p(s-s')-2} dy dx \right) \frac{|f(\sigma) - f(\tau)|^p}{(\tau - \sigma)^{1+ps}} d\tau d\sigma \\
& \leq C \int_{0 \leq \sigma \leq \tau \leq 1} \frac{|f(\sigma) - f(\tau)|^p}{(\tau - \sigma)^{1+ps}} d\tau d\sigma.
\end{aligned}$$

The constant C is determined by

$$\int_0^\sigma \int_\tau^1 (y-x)^{\delta-2} dy dx \leq \int_0^\sigma \int_\sigma^1 (y-x)^{\delta-2} dy dx \leq \begin{cases} \frac{1-2^{1-\delta}}{\delta(\delta-1)}, & \text{if } \delta = p(s-s') \neq 1 \\ \ln(2), & \text{if } \delta = p(s-s') = 1 \end{cases}.$$

Making use of Almgren's bilipschitz embedding ξ we deduce that (B.5) holds as well for multivalued functions i.e. for any $u \in W^{s,p}([0, 1], \mathcal{A}_Q(\mathbb{R}^n))$

$$\text{(B.7)} \quad \int_{0 \leq x \leq y \leq 1} \frac{\max_{\sigma, \tau \in [x, y]} \mathcal{G}(u(\sigma), u(\tau))^p}{|x - y|^{1+ps'}} dx dy \leq C \int_{0 \leq \sigma \leq \tau \leq 1} \frac{\mathcal{G}(u(\sigma), u(\tau))^p}{|\sigma - \tau|^{1+ps}} d\sigma d\tau.$$

We observed $W^{s,p}([0, 1], \mathcal{A}_Q(\mathbb{R}^n)) \subset C^{0, s-\frac{1}{p}}([0, 1], \mathcal{A}_Q(\mathbb{R}^n))$, so that we may apply the theory of regular selections developed in [11]. Especially we use the proof of [11, theorem 1.1]. For a given $u \in W^{s,p}([0, 1], \mathcal{A}_Q(\mathbb{R}^n))$ we can find $v = (v_1, \dots, v_Q) : [0, 1] \rightarrow (\mathbb{R}^n)^Q$ continuous with the property that $[v(t)] = \sum_{i=1}^Q \llbracket v_i(t) \rrbracket = u(t)$ on $[0, 1]$ and there is a constant $C_Q > 0$ s.t. for any $0 \leq x \leq y \leq 1$

$$|v(x) - v(y)| \leq C_Q \max_{\sigma, \tau \in [x, y]} \mathcal{G}(u(\sigma), u(\tau)).$$

Combining this with (B.7) gives the remaining part (ii) of the lemma. \square

APPENDIX C. CONSTRUCTION OF BILIPSCHITZ MAPS BETWEEN B_{1+} AND $\Omega_F \cap B_1$

Before showing the general situation, $\Omega_F \cap B_1$ with $\Omega_F = \{(x', x_N) \in \mathbb{R}^N : x_N > F(x')\}$, $F \in C^1(\mathbb{R}^{N-1})$, we consider the similar case of a bilipschitz map between B_1 and the upper half ball $B_{1+} = B_1 \cap \{x_N > 0\}$ that preserves "radial" homogeneity.

It is of interest for us to preserve "radial" homogeneity in the context of constructing competitors. We want to make use of the interpolation lemma on annuli, lemma A.9. We cannot use a generic bilipschitz map between B_1 and B_{1+} , because in general it is not true that if $G : U \rightarrow V$ is bilipschitz and $\psi_k : U \rightarrow U$ a sequence of diffeomorphisms that satisfy $\psi_k \rightarrow id$ then $G \circ \psi_k \circ G^{-1} \rightarrow 1$ with $Lip(G \circ \psi_k \circ G^{-1}) \rightarrow 1$ as $k \rightarrow \infty$.

Lemma C.1. *There is a bilipschitz map $G : \overline{B_1} \rightarrow \overline{B_{1+}}$ that preserves "radial" homogeneity in the sense that*

$$G \circ \frac{1}{R} \circ G^{-1}(y) = \left(1 - \frac{1}{R}\right)c + \frac{1}{R}y;$$

where $c = \frac{e_N}{2} = (0, \dots, 0, \frac{1}{2})$ and $0 < R$.

Proof. We make the ansatz $G(x) = c + s(\hat{x})x$ for a piecewise C^1 function $s : \mathcal{S}^{N-1} \rightarrow \partial B_{1+}$ with bounded derivative, where $\hat{x} = \frac{x}{|x|}$. The constraints $|c + s(x)x|^2 = 1$ for $x \in \mathcal{S}^{N-1} \cap \{x_N \geq a\}$ and $\langle e_N, c + s(x)x \rangle = 0$ for $x \in \mathcal{S}^{N-1} \cap \{x_N \leq a\}$ for some $-1 < a < 0$ determine s and a uniquely to $a = -\frac{1}{\sqrt{5}}$ and

$$s(x) = s(x_N) = \begin{cases} \frac{1}{2} \left(-x_N + \sqrt{x_N^2 + 3} \right), & \text{if } x_N \geq -\frac{1}{\sqrt{5}} \\ -\frac{1}{2x_N}, & \text{if } x_N \leq -\frac{1}{\sqrt{5}}. \end{cases}$$

The derivative is

$$s'(x_N) = \begin{cases} -\frac{1}{2} \left(1 - \frac{x_N}{\sqrt{x_N^2 + 3}} \right), & \text{if } x_N > -\frac{1}{\sqrt{5}} \\ \frac{1}{2x_N^2}, & \text{if } x_N < -\frac{1}{\sqrt{5}}; \end{cases}$$

So we may check the bounds $|s'| < 3$ and $\frac{1}{2} \leq s(x_N) \leq \frac{\sqrt{5}}{2}$. Furthermore we got $\text{grad } s(x) = \text{grad}_{\mathcal{S}^{N-1}} s(x) = s'(x_N)(\mathbf{1} - x \otimes x)e_N$. The inverse is explicitly given by $G^{-1}(y) = \frac{1}{s(\widehat{y-c})}(y - c)$. We got that G and G^{-1} are almost everywhere C^1 with derivatives

$$\begin{aligned} DG(x) &= s(\hat{x}) \mathbf{1} + \hat{x} \otimes \text{grad } s(\hat{x}) \\ DG^{-1}(y) &= \frac{1}{s(\widehat{y-c})} \mathbf{1} - \widehat{y-c} \otimes \frac{\text{grad } s(\widehat{y-c})}{s^2(\widehat{y-c})}. \end{aligned}$$

The "radial" homogeneity follows i.e. $G \circ \frac{1}{R} \circ G^{-1}(y) = G(\frac{1}{s(\widehat{y-c})} \frac{y-c}{R}) = (1 - \frac{1}{R})c + \frac{1}{R}y$. Therefore $DG \circ \frac{1}{R} \circ G^{-1} = \frac{1}{R} \mathbf{1}$ converging to $\mathbf{1}$ as $R \rightarrow 1$. \square

Lemma C.2. *For any $F \in C^1(\mathbb{R}^{N-1})$ that satisfies $F(0) = 0$, $\text{grad } F(0) = 0$ and $\|\text{grad } F\|_\infty < \frac{1}{4}$ there exists a C^1 -diffeomorphism*

$$G_F : \overline{B_{1+}} \rightarrow \overline{\Omega_F \cap B_1}$$

with bounds $\|DG_F - \mathbf{1}\|_\infty, \|DG_F^{-1} - \mathbf{1}\|_\infty < 10 \|\text{grad } F\|_\infty$.

Furthermore if F_k is a sequence of admissible maps with $F_k \rightarrow F$ in C^1 then $G_{F_k} \rightarrow G_F$ in C^1 .

Proof. Let F be fixed, then $\psi : (x', x_N) \mapsto (x', x_N + F(x'))$ is a C^1 -diffeomorphism between \mathbb{R}_+^N and Ω_F . Its inverse is $\psi^{-1}(x', x_N) = (x', x_N - F(x'))$. We make again an ansatz for $G = G_F$. Set $G(x) = \psi(s(\hat{x})x)$ where $s : \mathcal{S}^{N-1} \rightarrow \mathbb{R}_+$ satisfies $\psi(s(y)y) \in \Omega_F \cap \mathcal{S}^{N-1}$ for all $y \in \mathcal{S}_+^{N-1}$. The inverse for such a G is $G^{-1}(x) = \frac{1}{s(\widehat{\psi^{-1}(x)})} \psi^{-1}(x)$.

As a consequence of the implicit function theorem applied to the level set at 1 of the auxiliary function

$$h(y, s) = |\psi(sy)|^2,$$

$s \in C^1(\mathcal{S}_+^{N-1}, \mathbb{R}_+)$ has the desired properties. Note that $s(e_N) = 1$ because $h(e_N, 1) = 1$.

Existence: to every $y \in \mathcal{S}_+^{N-1}$ there exists $s(y) \in \mathbb{R}_+$ s.t. $h(y, s(y)) = 1$ and $1 - \|\text{grad } F\|_\infty \leq \frac{1}{s} \leq 1 + \|\text{grad } F\|_\infty$, because

$$\begin{aligned} h(y, s) &= s^2 \left| y + \frac{F(sy')}{s} e_N \right|^2 \\ &\leq s^2 (1 + \|\text{grad } F\|_\infty)^2 < 1 \text{ if } s < \frac{1}{1 + \|\text{grad } F\|_\infty} \\ &\geq s^2 (1 - \|\text{grad } F\|_\infty)^2 > 1 \text{ if } s > \frac{1}{1 - \|\text{grad } F\|_\infty}. \end{aligned}$$

C_{loc}^1 *homeomorphism:* every tuple (y_0, s_0) with $h(y_0, s_0) = 1$ has a neighbourhood $U \times I$ in $\mathcal{S}_+^{N-1} \times \mathbb{R}_+$ and a C^1 map $s : U \rightarrow I$, C^1 with $h(y, s(y)) = 1$ on U . This follows from the implicit function theorem, because at $x_0 = s_0 y_0$

$$\begin{aligned} \frac{1}{2} s \frac{\partial h}{\partial s} &= 1 - \langle \psi(x_0), \psi(x_0) - d\psi(x_0)x_0 \rangle \\ &= 1 - \psi_N(x_0) (F(x'_0) - \langle \text{grad } F(x'_0), x'_0 \rangle) \geq 1 - 2 \|\text{grad } F\|_\infty \geq \frac{1}{2}. \end{aligned}$$

Uniqueness/ well-definition: this is a consequence of $\frac{\partial h}{\partial s} > 0$ for each such tuple (y_0, s_0) , so there cannot be two $s_1 < s_2$ with $h(y_0, s_1) = 1 = h(y_0, s_2)$.

Bounds on $\text{grad } s = \text{grad}_{\mathcal{S}^{N-1}} s$: Fix any generic $\tau \in T_y \mathcal{S}^{N-1}$ and so $0 = (D_\tau h + \frac{\partial h}{\partial s} D_\tau s)(y, s(y))$. Furthermore writing $x = s(y)y$ we have

$$\frac{1}{2s} D_\tau h(y, s) = \frac{1}{s} \langle \psi(x), d\psi(x)s\tau \rangle = \tau_N F(x') + \psi_N(x') \langle \text{grad } F(x'), \tau' \rangle,$$

that gives

$$\left| \frac{1}{2s} D_\tau h(y, s) \right| \leq \sqrt{2} \|\text{grad } F\|_\infty.$$

We conclude

$$|D_\tau s(y)| = s^2 \frac{\left| \frac{1}{2s} D_\tau h \right|}{\left| \frac{1}{2} s \frac{\partial h}{\partial s} \right|} \leq 3s^2 \|\text{grad } F\|_\infty \leq 16 \|\text{grad } F\|_\infty.$$

Bounds on DG, DG^{-1} : One calculates explicitly that

$$\begin{aligned} DG(x) &= d\psi(s(\widehat{x})x) (s(\widehat{x})\mathbf{1} + \widehat{x} \otimes \text{grad } s(\widehat{x})) \\ &= s(\widehat{x})\mathbf{1} + \widehat{x} \otimes \text{grad } s(\widehat{x}) + (e_N \otimes \text{grad } F) (s(\widehat{x})\mathbf{1} + \widehat{x} \otimes \text{grad } s(\widehat{x})). \end{aligned}$$

As we have seen $|s(\widehat{x}) - 1| \leq \frac{\|\text{grad } F\|_\infty}{1 - \|\text{grad } F\|_\infty}$. Combining all obtained bounds one can conclude $\|DG(x) - \mathbf{1}\|_\infty \leq 10 \|\text{grad } F\|_\infty$. DG^{-1} is given explicitly by

$$\begin{aligned} DG^{-1}(x) &= \frac{1}{s(\widehat{\psi^{-1}(x)})} d\psi^{-1}(x) - \widehat{\psi^{-1}(x)} \otimes \frac{\text{grad } s(\widehat{\psi^{-1}(x)})}{s^2(\widehat{\psi^{-1}(x)})} \\ &= \frac{1}{s(\widehat{\psi^{-1}(x)})} \mathbf{1} - \frac{1}{s(\widehat{\psi^{-1}(x)})} e_N \otimes \text{grad } F - \widehat{\psi^{-1}(x)} \otimes \frac{\text{grad } s(\widehat{\psi^{-1}(x)})}{s^2(\widehat{\psi^{-1}(x)})}. \end{aligned}$$

Combing as before all obtained bounds especially $\left| \frac{1}{s(\widehat{\psi^{-1}(x)})} - 1 \right| \leq \|\text{grad } F\|_\infty$ one can get $\|DG^{-1}(x) - \mathbf{1}\|_\infty \leq 6 \|\text{grad } F\|_\infty$.

The convergence statement follows as a consequence of the implicit function theorem, because $F_k \rightarrow F$ in C^1 then implies $s_{F_k} \rightarrow s_F$ in C^1 . \square

Part 2. Examples of holomorphic functions vanishing to infinite order at the boundary

2.4. INTRODUCTION

In general branching phenomena are of interest in geometric measure theory and geometry, and are strongly related to vanishing phenomena in the context of PDE's. There is some literature on branching in the interior and one has unique continuation results for PDE's in the interior of their domains of definition. Little seems to be known towards the boundary. This part presents examples of holomorphic functions that vanish to infinite order at points at the boundary of their domain of definition. Thereafter we discuss some implication in the context of minimal surfaces, Q -valued functions and unique continuation. These might be an invitation and motivation to the study on boundary behaviour.

Let me shortly explain how I got motivated to this approach, looking for holomorphic functions vanishing to infinite order with a "large" zero set.

My own attempts trying to understand the boundary regularity of Q -valued Dirichlet minimizing imposed the question: "Can one say something about the structure of the singular set towards the boundary?"

Almgren's frequency function is a key tool to study the singular set in the interior. It is monotone quantity that enables a stratification procedure, compare for example [12, section 3.4 - 3.6] or the work of N. Wickramasekera et al. Such a stratification procedure built on a monotone quantity had been successfully applied as well in other context. (In some sense they can be considered refinements of the "dimension reducing" argument of Federer [6].) Unfortunately Almgren's frequency function is only monotone in the interior, so a direct extension to the boundary is not possible.

An inspiring discussion with N. Wickramasekera about possible expectations about the structure of the singular set towards the boundary made it apparent that a first impression could be obtained by looking at harmonic or holomorphic functions with zeros accumulating towards the boundary. This link was motivated by the fact that Almgren's frequency functions has been successfully applied in the context of unique continuation (e.g. [7]) where the vanishing order is measured with the frequency function. The example of this part are perhaps of interest in other context such as minimal surfaces and unique continuation. This is discussed in more detail in section 2.7.

To give a first impression we state here an implication to Q -valued Dirichlet minimizers heuristically. We avoid introducing some terminology and the precise statement is corollary 2.7.5.

Corollary*:

Given $0 < s \leq 1$, an integer $Q \geq 2$ there is a Q -valued function u , Dirichlet minimizing with respect to compact perturbations satisfying the additional properties:

- (i) *the trace $u|_{\partial\mathbb{R}_+^2}$ is "smooth";*
- (ii) *if $s < 1$ then $\mathcal{H}^s(\overline{\text{sing}(u)}) = 1$ and if $s = 1$ then $\dim_{\mathcal{H}}(\overline{\text{sing}(u)}) = 1$.*

But now let us state the underlying properties of the holomorphic functions. We present examples of holomorphic functions on the half plane $\mathbb{C}_+ = \{z \in \mathbb{C}: \Re(z) > 0\}$ that admit C^∞ -extension to $\overline{\mathbb{C}_+}$ and vanish to infinite order at boundary points. Their properties are:

Lemma 2.4.1. *Let $0 < s \leq 1$ be given. There exist*

- (i) a nowhere dense compact Cantor type subset $E_s \subset [0, 1]$ with $\mathcal{H}^s(E_s) = 1$ if $0 < s < 1$ and $\dim_{\mathcal{H}}(E_1) = 1$;
- (ii) holomorphic functions $F(z)$, $G(z)$ on \mathbb{C}_+ with the property that $f(z) = e^{-F(z)}$, $g(z) = G(z)e^{-F(z)}$ admit C^∞ -extensions to $\overline{\mathbb{C}_+}$. Moreover, f, g vanish to infinite order at any $z \in -iE_s$ and for every $z \in -iE_s$ there is a sequence $z_k \in \mathbb{C}_+$ with $z_k \rightarrow z$ and $g(z_k) = 0$ for all k .

The functions are constructed similar to the Weierstrass' function, an example of a non-differentiable function. Instead of an infinite series we use infinite products of the following holomorphic *building blocks*:

$$(2.4.1) \quad \begin{aligned} a(z) &= e^{-z^{-\alpha}} \text{ for } 0 < \alpha < 1 \\ b(z) &= \cos(\ln(z))e^{-z^{-\alpha}} \text{ for } 0 < \alpha < 1. \end{aligned}$$

The results of sections 2.5 and 2.6 combined prove lemma 2.4.1. The section 2.7 presents some first implications to branching of minimal surfaces, Q -valued functions and unique continuation.

2.5. CONSTRUCTION AND PROPERTIES OF THE SET E_s

The construction is a classical Cantor type construction. Nonetheless for the sake of completeness and to fix certain parameters we present the construction in detail. We follow closely an approach of Falconer in [5, Theorem 1.15].

Lemma 2.5.1. *Let $0 < s \leq 1$ be given. Then there is a nowhere dense compact subset $E_s \subset [0, 1]$ s.t. $\mathcal{H}^s(E_s) = 1$ if $0 < s < 1$ and $\dim(E_1) = 1$.*

Proof. The set E_s is obtained classically as the intersection of a decreasing family of compact sets

$$E_s = \bigcap_{k=1}^{\infty} \bigcup_{l=1}^{2^k} E_{k,l}.$$

The compact subintervals $E_{k,l}$ are defined inductively.

We fix a sequence of parameters by

$$\frac{1}{\sigma_k} = \begin{cases} \frac{1}{s}, & \text{if } 0 < s < 1 \\ 1 + k^{\frac{2}{3}} - (k-1)^{\frac{2}{3}}, & \text{if } s = 1 \end{cases}$$

In both cases we have $\sigma_k \leq \sigma_{k+1}$ for all k . If $s = 1$ we have $\frac{1}{\sigma_{k+1}} - \frac{1}{\sigma_k} = (k+1)^{\frac{2}{3}} - (k-1)^{\frac{2}{3}} - 2k^{\frac{2}{3}} < 0$ due to concavity of $t \mapsto t^{\frac{2}{3}}$.

We choose $E_{0,1} = [0, 1]$ and proceed inductively. Suppose $E_{k-1,l}$, $l = 1, \dots, 2^{k-1}$ defined, then $E_{k,2l-1}, E_{k,2l}$ are the closed subintervals obtained by removing an open interval in the middle of $E_{k-1,l}$ with

$$(2.5.1) \quad |E_{k,2l-1}|^{\sigma_k} = |E_{k,2l}|^{\sigma_k} = \frac{1}{2} |E_{k-1,l}|^{\sigma_k}.$$

We obtained 2^k closed intervals $E_{k,l}$ of equal length

$$(2.5.2) \quad |E_{k,l}| = 2^{-\frac{1}{\sigma_k}} |E_{k-1,l'}| = \begin{cases} 2^{-\frac{k}{s}}, & \text{if } 0 < s < 1 \\ 2^{-k-k^{\frac{2}{3}}}, & \text{if } s = 1 \end{cases}$$

where we used that $\sum_{l=1}^k \sigma_k^{-1} = \frac{k}{s}$ if $0 < s < 1$ and $\sum_{l=1}^k \sigma_k^{-1} = k + k^{\frac{2}{3}}$ if $s = 1$.

In a first step we will check that $\mathcal{H}^s(E_s) \leq 1$ ($\mathcal{H}^1(E_1) = 0$). To do so, let $\delta > 0$ be given. Due to (2.5.2) there is $k_0 > 0$ with $|E_{k_0,l}| < \delta$. Hence $\{E_{k,l}\}_{l=1}^{2^k}$ is an

admissible δ -cover for E_s for any $k \geq k_0$. With (2.5.2) in mind we have

$$(2.5.3) \quad \mathcal{H}_\delta^s(E_s) \leq \sum_{l=1}^{2^k} |E_{k,l}|^s = \begin{cases} 2^k \left(2^{-\frac{k}{s}}\right)^s = 1, & \text{if } 0 < s < 1 \\ 2^k 2^{-k-\frac{2}{3}} \rightarrow 0, & \text{if } s = 1, k \rightarrow \infty. \end{cases}$$

Now in the second step we check that $\mathcal{H}^s(E_s) \geq 1$ if $s < 1$ and $\mathcal{H}^\sigma(E_1) = +\infty$ for all $\sigma < 1$ if $s = 1$. Equivalently we have to show that for any $\epsilon > 0$ there is a $\delta > 0$ with the property that for any δ -cover \mathcal{U} of E_s we have

$$(2.5.4) \quad \begin{aligned} \sum_{C \in \mathcal{U}} \text{diam}(C)^s &\geq \mathcal{H}_\delta^s(E_s) > 1 - \epsilon, & \text{if } 0 < s < 1 \\ \sum_{C \in \mathcal{U}} \text{diam}(C)^\sigma &\geq \mathcal{H}_\delta^\sigma(E_1) > \frac{1}{\epsilon}. & \text{if } s = 1 \text{ i.e. } \sigma < 1 \end{aligned}$$

Let $\epsilon > 0$, $\sigma < 1$ be given. We fix $k_0 > 0$ large, determined later s.t. at least $\sigma_{k_0} > \sigma$ and $0 < \delta < |E_{k_0,l}|$.

Fix an admissible δ -cover \mathcal{U} by intervals $E_{k,l}$. Hence $k > k_0$ for any of these intervals. The compact intervals $E_{k,l}$ are relative open to the compact set E_s , so that the cover can be assumed to be finite. Removing all intervals that are contained in some other of the collection we can even assume that they are mutually disjoint. Let $E_{k,2l-1}$ (or $E_{k,2l}$) be one of the shortest intervals in \mathcal{U} . Its companion $E_{k,2l}$ (respectively $E_{k,2l-1}$) has to be in \mathcal{U} as well because all intervals are disjoint and they are one of the shortest. The sums in (2.5.4) do not increase if we replace these two intervals by its predecessor $E_{k-1,l} \supset E_{k,2l-1} \cup E_{k,2l}$ because

$$\begin{aligned} |E_{k,2l-1}|^s + |E_{k,2l}|^s &= |E_{k-1,l}|^s, & \text{if } 0 < s < 1 \\ |E_{k,2l-1}|^\sigma + |E_{k,2l}|^\sigma &= 2^{1-\frac{\sigma}{s_k}} |E_{k-1,l}|^\sigma \geq |E_{k-1,l}|^\sigma, & \text{if } s = 1 \text{ i.e. } \sigma < 1 \end{aligned}$$

where we used (2.5.1) and $\sigma_k \geq \sigma_{k_0} > \sigma$. We may proceed in this way, replacing the shortest intervals by larger ones without increasing the value of the sums, until we reach that all intervals are of same size i.e. $\mathcal{U} \rightarrow \{E_{k_1,l}\}_{l=1}^{2^{k_1}}$ for some $k_1 > k_0$. We conclude

$$\begin{aligned} \sum_{C \in \mathcal{U}} \text{diam}(C)^s &\geq \sum_{l=1}^{2^{k_1}} |E_{k_1,l}|^s = 1, & \text{if } 0 < s < 1 \\ \sum_{C \in \mathcal{U}} \text{diam}(C)^\sigma &\geq \sum_{l=1}^{2^{k_1}} |E_{k_1,l}|^\sigma = 2^{(1-\sigma)k_1 - \sigma k_1^{\frac{2}{3}}} > \frac{1}{\epsilon}. & \text{if } s = 1 \text{ i.e. } \sigma < 1 \end{aligned}$$

where we used (2.5.3) and $k_1 > k_0$ with $k_0 > 0$ sufficient large s.t. $2^{(1-\sigma)k_0 - \sigma k_0^{\frac{2}{3}}} > \frac{1}{\epsilon}$.

It remains to argue that the assumption that the δ -cover is made out of intervals $E_{k,l}$ is no real restriction. Fix any δ -cover \mathcal{V} . We can assume that it consists of open intervals without changing the value in (2.5.4) significantly. Since E_s is compact the cover can be assumed to be finite.

Firstly let us argue for E_1 . Any interval $I \in \mathcal{V}$ intersects at most three intervals $E_{k_I,l}$ with $|E_{k_I,l}| \leq |I| < |E_{k_I-1,l}|$. Otherwise I would need to contain an interval of length at least $|E_{k_I-1,l}|$ due to the Cantor type construction. This is impossible by the choice of k_I . Replacing I by these at most three intervals $E_{k_I,l}$, and the same

for any other interval in \mathcal{I} we obtain an open cover \mathcal{U} by intervals $E_{k,l}$. Furthermore

$$\sum_{E_{k,l} \in \mathcal{U}} |E_{k,l}|^\sigma \leq 3 \sum_{I \in \mathcal{V}} |I|^\sigma.$$

We had just shown that the left hand side is larger then $\frac{1}{\epsilon}$, so (2.5.4) holds for $s = 1$.

If $0 < s < 1$ we transform the δ -cover \mathcal{V} iteratively without increasing the sum in (2.5.4) to a δ -cover \mathcal{U} by sets in $E_{k,l}$. At first contracting each interval $I \in \mathcal{V}$ we pass to a cover \mathcal{V}_1 by closed intervals J with endpoints that are the endpoints of some $E_{k,l}$. This process ensures $\sum_{I \in \mathcal{V}} |I|^s \geq \sum_{J \in \mathcal{V}_1} |J|^s$. Let J be any such closed interval in the cover and $J \subset E_{k-1,l}$ for some k, l . Then

$$(2.5.5) \quad |J \cap E_{k,2l-1}|^s + |J \cap E_{k,2l}|^s \leq |J \cap E_{k-1,l}|^s$$

because $|E_{k,2l-1}|^s + |E_{k,2l}|^s = |E_{k-1,l}|^s$ and the left-hand side of (2.5.5) increases faster then the right-hand side. If either $J \cap E_{k,2l-1} \neq E_{k,2l-1}$ or $J \cap E_{k,2l} \neq E_{k,2l}$, we repeat the process, replacing $J \cap E_{k,2l-1}$ and $J \cap E_{k,2l}$ by smaller intervals. This process terminates after finitely many steps till we reach the desired cover \mathcal{U} . By construction we ensured $\sum_{J \in \mathcal{V}_1} |J|^s \geq \sum_{E_{k,l} \in \mathcal{U}} |E_{k,l}|^s = 1$. This proves (2.5.4) if $0 < s < 1$. □

2.6. CONSTRUCTION OF THE HOLOMORPHIC FUNCTIONS

The Cantor set E_s was obtained as

$$E_s = \bigcap_{k=1}^{\infty} \bigcup_{l=1}^{2^k} E_{k,l}.$$

Based on this construction, we define the index set:

$$\mathcal{I} = \{(k, l) : k = 1, \dots, \infty, l = 1, \dots, 2^k\} \text{ with } \tau = (k, l) \in \mathcal{I}.$$

Recall that the enumeration had been chosen s.t. $E_{k,2l-1} \cup E_{k,2l} \subset E_{k-1,l} \forall (k, l)$. The Cantor set E_s constructed in lemma 2.5.1, i.e. (2.5.2), had the property that

$$|E_\tau| = |E_{k,l}| = \begin{cases} 2^{-\frac{k}{s}}, & \text{if } 0 < s < 1 \\ 2^{-k-k\frac{2}{3}}, & \text{if } s = 1 \end{cases} \quad \forall \tau \in \mathcal{I}.$$

We denote with y_τ the left boundary point of the compact interval E_τ . Furthermore it is useful to fix some terminology. $\mathbb{R}_- = \{z = x + i0 : x < 0\}$ denotes the negative real axis. We will use $z + iy_\tau = r_\tau e^{i\theta_\tau}$ for any $\tau \in \mathcal{I}$. And for any $y \in \mathbb{R}$ let $\mathbb{R}_- iy$ be the by $-iy$ translated negative real axis i.e. the set $\{x - iy : x < 0\}$. And we will use

$$\mathbb{R}_- - iE_s = \bigcup_{y \in E_s} (\mathbb{R}_- - iy) = \{x - iy : x \in \mathbb{R}_-, y \in E_s\}.$$

The proof to lemma 2.4.1 is split into two parts. In the next paragraph we construct holomorphic functions F, G based on the Cantor set E_s and then in the subsequent paragraph the C^∞ extension is proven.

2.6.1. Holomorphy. On the slit plane $\mathbb{C} \setminus \mathbb{R}_-$ the principal value of the logarithmic function $\ln : \mathbb{C} \setminus \mathbb{R}_- \rightarrow \mathbb{C} \cap \{-\pi < \Im(z) < \pi\}$ is single valued and holomorphic. So will be all roots for $\alpha \in \mathbb{R}$ defined as $z^\alpha = e^{\alpha \ln(z)}$.

As composition of holomorphic functions on $\mathbb{C} \setminus \mathbb{R}_-$ the building blocks, $a(z) = e^{-z^{-\alpha}}$, $b(z) = \cosh(\ln(z))e^{-z^{-\alpha}}$ are clearly holomorphic on $\mathbb{C} \setminus \mathbb{R}_-$. $(z + iy_\tau)^{-\alpha_k} = r_\tau e^{-i\alpha\theta_\tau}$ is single valued and holomorphic on $\mathbb{C} \setminus (\mathbb{R}_- - iy_\tau) \subset \mathbb{C} \setminus (\mathbb{R}_- - iE_s)$ for every $\tau \in \mathcal{I}$, $\alpha_k \in \mathbb{R}$.

Lemma 2.6.1. *Given a sequence of complex numbers $a_k \in \mathbb{C}$ with $\sum_{k=0}^{\infty} 2^k |a_k| < \infty$ and a sequence of real numbers $0 < \alpha_k \leq 1$ then*

$$F(z) = \sum_{\tau \in \mathcal{I}} a_k (z + iy_{\tau})^{-\alpha_k}$$

is holomorphic on $\mathbb{C} \setminus \{\mathbb{R}_- - iE_s\}$ and so is $e^{-F(z)}$.

Proof. For a fixed $0 < d < 1$ we have for any $z \in \{z \in C : \text{dist}(z, -iE_s) > d\}$ satisfies $|(z + iy_{\tau})^{-\alpha_k}| = r_{\tau}^{-\alpha_k} \leq d^{-1}$. So that the sum $\sum_{\tau \in \mathcal{I}} |a_k (z + iy_{\tau})^{-\alpha_k}| \leq d^{-1} \sum_{k=1}^{\infty} 2^k |a_k| < \infty$ converges absolutely. F is therefore the uniform limit of holomorphic functions on $\{z \in C : \text{dist}(z, -iE_s) > d\}$ and so itself holomorphic. d has been arbitrary and therefore F is holomorphic on $\mathbb{C} \setminus (\mathbb{R}_- - iE_s)$. $e^{-F(z)}$ is the composition of two holomorphic functions and so itself holomorphic on the same set. \square

Lemma 2.6.2. *Given a sequence of non-negative real numbers $b_k \in \mathbb{R}_+$ that satisfies $\sum_{k=0}^{\infty} 2^k b_k < \infty$, then for any subset $\mathcal{J} \subset \mathcal{I}$*

$$(2.6.1) \quad G_{\mathcal{J}}(z) = \prod_{\tau \in \mathcal{J}} \cos(b_k \ln(z + iy_{\tau}))$$

is holomorphic on $\mathbb{C} \setminus (\mathbb{R}_- - iE_s)$ and uniformly bounded by

$$|G_{\mathcal{J}}(z)| \leq e^{\sum_{\tau \in \mathcal{J}} b_k |\theta_{\tau}|} \leq e^{\pi \sum_{k=0}^{\infty} 2^k b_k}.$$

Proof. As a composition of holomorphic functions $\cos(b_k \ln(z + iy_{\tau}))$ is holomorphic on $\mathbb{C} \setminus (\mathbb{R}_- - iE_s)$ for every $\tau \in \mathcal{I}$. Since $\cos(x + iy) = \cos(x) \cosh(y) - i \sin(x) \sinh(y)$ we have $\cos(b_k \ln(z + iy_{\tau})) = \cos(b_k \ln(r_{\tau})) \cosh(b_k \theta_{\tau}) + i \sin(-b_k \ln(r_{\tau})) \sinh(b_k \theta_{\tau})$. So we got that for every $\tau \in \mathcal{I}$

$$(2.6.2) \quad |\cos(b_k \ln(r_{\tau})) \cosh(b_k \theta_{\tau})| \leq |\cos(b_k \ln(z + iy_{\tau}))| \leq \cosh(b_k \theta_{\tau})$$

$$(2.6.3) \quad \frac{\Im(\cos(b_k \ln(z + iy_{\tau})))}{\Re(\cos(b_k \ln(z + iy_{\tau})))} = \tan(-b_k \ln(r_{\tau})) \tanh(b_k \theta_{\tau}).$$

To show that (2.6.1) is well defined and holomorphic, fix $0 < d < 1$ and $k_0 \in \mathbb{N}$ sufficient large s.t. $0 \leq -b_k \ln(d) \leq \frac{\pi}{4}$ for all $k \geq k_0$. This ensures that for any $z \in \{d < \text{dist}(z, -iE_s) < \frac{1}{d}\}$ and $\tau \in \mathcal{I} \cap \{k \geq k_0\}$ we have $-\frac{\pi}{4} < b_k \ln(r_{\tau}) < \frac{\pi}{4}$. Hence $\ln(\cos(b_k \ln(z + iy_{\tau})))$ is a holomorphic function on $\{d < \text{dist}(z, -iE_s) < \frac{1}{d}\}$ if $\tau \in \mathcal{I} \cap \{k \geq k_0\}$.

(2.6.4)

$$\ln(\cosh(b_k \theta_{\tau})) + \ln(\cos(b_k \ln(r_{\tau}))) \leq \ln(|\cos(b_k \ln(z + iy_{\tau}))|) \leq \ln(\cosh(b_k \theta_{\tau}))$$

where we used (2.6.2). This is the real part of $\ln(\cos(b_k \ln(z + iy_{\tau})))$. Its imaginary part, the argument of $\cos(b_k \ln(z + iy_{\tau}))$ can be estimated by $|b_k \ln(r_{\tau})|$ taking into account that $|\tanh| < 1$ and $-\frac{\pi}{4} < b_k \ln(r_{\tau}) < \frac{\pi}{4}$. Combining both we deduce

$$|\ln(|\cos(b_k \ln(z + iy_{\tau}))|)| \leq \ln(\cosh(b_k \theta_{\tau})) - \ln(\cos(b_k \ln(r_{\tau}))) + |b_k \ln(r_{\tau})|.$$

One checks that $h(x) = \frac{-\ln(\cos(x))}{x}$ is monotone increasing on $]0, \frac{\pi}{2}[$, hence for $|x| \leq \frac{\pi}{4}$ we have $-\ln(\cos(x)) \leq C|x|$ with $C = h(\frac{\pi}{4})$. Consequently we have

$$|\ln(|\cos(b_k \ln(z + iy_{\tau}))|)| \leq b_k |\theta_{\tau}| + (C + 1)|b_k \ln(r_{\tau})| \leq (\pi - \ln(d)(C + 1))b_k;$$

$\sum_{\tau \in \mathcal{I} \cap \{k \geq k_0\}} |\ln(\cos(b_k \ln(z + iy_{\tau})))| < (\pi - \ln(d)(C + 1)) \sum_{k=k_0}^{\infty} 2^k b_k$ converges uniformly on $\{d < \text{dist}(z, -iE_s) < \frac{1}{d}\}$ so that

$$G_1(z) = e^{\sum_{\tau \in \mathcal{J}, k \geq k_0} \ln(\cos(b_k \ln(z + iy_{\tau})))}$$

is holomorphic on $\{d < \text{dist}(z, -iE_s) < \frac{1}{d}\}$. (2.6.4) showed that $\Re(\ln(\cos(b_k \ln(z + iy_\tau)))) \leq \ln(\cosh(b_k \theta_\tau)) \leq b_k |\theta_\tau|$ and therefore

$$|G_1(z)| = e^{\sum_{\tau \in \mathcal{J}, k \geq k_0} \Re(\ln(\cos(b_k \ln(z + iy_\tau))))} \leq e^{\sum_{\tau \in \mathcal{J}, k \geq k_0} b_k |\theta_\tau|}.$$

$$G_2(z) = \prod_{\substack{\tau \in \mathcal{J} \\ k < k_0}} \cos(b_k \ln(z + iy_\tau))$$

is the product of finitely many holomorphic functions on $\mathbb{C} \setminus (\mathbb{R}_- - iE_s)$ and so itself holomorphic with

$$|G_2(z)| \leq \prod_{\substack{\tau \in \mathcal{J} \\ k < k_0}} |\cos(b_k \ln(z + iy_\tau))| \leq \prod_{\substack{\tau \in \mathcal{J} \\ k < k_0}} \cosh(b_k \theta_\tau) \leq e^{\sum_{\tau \in \mathcal{J}, k < k_0} b_k |\theta_\tau|}$$

where we used (2.6.2). Multiplication of G_1 and G_2 closes the argument. \square

$\cos(b_k \ln(z + iy_\tau)) = 0$ for $z = -iy_\tau + e^{-\frac{m\pi - \frac{\pi}{2}}{b_k}}$ for any $\tau = (k, l) \in \mathcal{I}$ and $m \in \mathbb{N}$, so that

$$G(z) = G_{\mathcal{I}}(z) = 0 \text{ for all } z = -iy_\tau + e^{-\frac{m\pi - \frac{\pi}{2}}{b_k}}, \tau = (k, l) \in \mathcal{I}, m \in \mathbb{N}.$$

Consequently we got the following:

Corollary 2.6.3. *Let α_k, a_k, b_k be sequences of non-negative real numbers, that satisfies $0 \leq \alpha_k \leq 1$ and $\sum_{k=1}^{\infty} 2^k a_k, \sum_{k=1}^{\infty} 2^k b_k < \infty$ then*

$$f(z) = e^{-F(z)}, \quad g(z) = G(z)e^{-F(z)}$$

are holomorphic on $\mathbb{C} \setminus (\mathbb{R}_- - iE_s)$. Furthermore

$$g(z) = 0 \text{ for } z = -iy_\tau + e^{-\frac{m\pi - \frac{\pi}{2}}{b_k}}, \tau = (k, l) \in \mathcal{I}, m \in \mathbb{N}.$$

2.6.2. C^∞ -extension. In this section we will show that one can choose sequences a_k, b_k, α_k appropriately (satisfying the conditions of corollary 2.6.3) such that f, g are holomorphic on \mathbb{C}_+ and admit a C^∞ -extension to $\overline{\mathbb{C}_+} = \{z \in \mathbb{C} : \Re(z) > 0\}$.

Firstly we check that the building blocks, a, b , introduced in (2.4.1), admit such a C^∞ -extension to $\overline{\mathbb{C}_+}$ and are vanishing to infinite order in 0 i.e.

$$(2.6.5) \quad \lim_{\substack{|z| \searrow 0 \\ z \in \mathbb{C}_+}} \left| \frac{d^m}{dz^m} a(z) \right|, \left| \frac{d^m}{dz^m} b(z) \right| = 0.$$

By induction one shows that there are constants $C = C(m), D = D(m) > 0$ and $\mu = \mu(m), \nu = \nu(m) \in \mathbb{R}$ (depending only on m) s.t. for any $0 < \alpha < 1$, $z = re^{i\theta} \in \mathbb{C} \setminus \mathbb{R}_-, r < 1$

$$\left| \frac{d^m}{dz^m} e^{-z^{-\alpha}} \right| \leq C |z^{-2m}| |e^{-z^{-\alpha}}| = Cr^{-2m} e^{-\Re(z^{-\alpha})}$$

and

$$\left| \frac{d^m}{dz^m} \cos(\ln(z)) \right| = \left| \mu \frac{\cos(\ln(z))}{z^m} + \nu \frac{\sin(\ln(z))}{z^m} \right| \leq Dr^{-m} \cosh(\theta).$$

Hence (2.6.5) holds if $r^{-m} e^{-\Re(z^{-\alpha})} \rightarrow 0$ as $r \rightarrow 0$ for every $m \in \mathbb{N}$. This is equivalent to $\Re(z^{-\alpha}) + m \ln(r) \rightarrow +\infty$ as $r \rightarrow 0$. For $z \in \overline{\mathbb{C}_+} \setminus \{0\}$ we have $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ and so

$$\Re(z^{-\alpha}) + m \ln(r) = r^{-\alpha} \cos(\alpha\theta) + m \ln(r) \geq r^{-\alpha} \cos(\alpha \frac{\pi}{2}) + m \ln(r) \rightarrow \infty (r \rightarrow 0).$$

Similarly we can conclude the extension for f, g :

Lemma 2.6.4. *Let the sequences be $a_k = b_k = \frac{2^{-k}}{k^2}$ and*

$$\alpha_k = \begin{cases} \alpha, & \text{if } 0 < s < 1 \text{ for some } s < \alpha < 1. \\ 1 - \frac{1}{2}k^{-\frac{1}{3}} \end{cases}$$

Then the function f, g of corollary 2.6.3 are holomorphic on $\mathbb{C} \setminus (\mathbb{R}_- - iE_s)$ and admit C^∞ extensions to $\overline{\mathbb{C}_+}$ with

$$\lim_{\substack{\text{dist}(z, -iE_s) \rightarrow 0 \\ z \in \overline{\mathbb{C}_+}}} \left| \frac{d^m}{dz^m} f(z) \right|, \left| \frac{d^m}{dz^m} g(z) \right| = 0.$$

Proof. That f, g are well-defined and holomorphic is the content of corollary 2.6.3. It remains to check the \mathbb{C}^∞ -extension.

Due to the general Leibnitz rule $\frac{d^m}{dz^m} f(z) = \sum_{n=0}^m \binom{m}{n} G^{(m-n)}(z) (e^{-F(z)})^{(n)}$ it is sufficient to check that for any $m, n \in \mathbb{N}$,

$$\lim_{\substack{\text{dist}(z, -iE_s) \rightarrow 0 \\ z \in \overline{\mathbb{C}_+}}} |G^{(m)}(z) (e^{-F(z)})^{(n)}| = 0.$$

Firstly we note that F is holomorphic on \mathbb{C}_+ , $(e^{-F(z)})' = -F'(z)e^{-F(z)}$ and

$$|F^{(m)}(z)| \leq \sum_{\tau \in \mathcal{I}} a_k \left| \frac{d^m}{dz^m} (z + iy_\tau)^{-\alpha_k} \right| \leq m! d^{-m-1} \sum_{k=1}^{\infty} a_k 2^k$$

for $z \in \mathbb{C}_+$, $\text{dist}(z, -iE_s) \geq d$, so that by induction we deduce

$$(2.6.6) \quad \left| \frac{d^m}{dz^m} e^{-F(z)} \right| \leq C d^{-m-1} |e^{-F(z)}| \text{ for } z \in \{\text{dist}(z, -iE_s) \geq d\}.$$

for a constant $C > 0$ that depends only on m and $\sum_{k=1}^{\infty} a_k 2^k = \frac{\pi^2}{6}$. Secondly, Cauchy's integral formula

$$G^{(m)}(z) = \frac{m!}{2\pi i} \oint_{\partial B_d(z)} \frac{\psi \circ G(w)}{(w-z)^{m+1}} dw$$

applies since G is holomorphic on $B_d(z)$. Combining it with the uniform bound on $|G|$ (lemma 2.6.2) gives

$$(2.6.7) \quad |G^{(m)}(z)| \leq \frac{m!}{d^m} \sup_{w \in B_d(z)} |G(w)| \leq \frac{Cm!}{d^m}.$$

Considering (2.6.6), (2.6.7) and the general Leibniz rule the C^∞ lemma follows if for every $m \in \mathbb{N}$

$$d^{-m} |e^{-F(z)}| = e^{-(\Re(F(z)) + m \ln(d))} \rightarrow 0 \text{ for } d = \text{dist}(z, -iE_s) \rightarrow 0.$$

This is equivalent to

$$(2.6.8) \quad \Re(F(z)) + m \ln(d) \rightarrow +\infty \text{ as } d \rightarrow 0.$$

To check it, let $z \in \overline{\mathbb{C}_+}$ with $d = \text{dist}(z, -iE_s) > 0$ be given. Fix $y \in E_s$ with $d = |z - iy|$ and $\tau_k = (k, l) \in \mathcal{I}$ with $y \in E_{\tau_k}$ for each $k \in \mathbb{N}$. Take $k_0 \in \mathbb{N}$ with

$$(2.6.9) \quad |E_{k_0+1, \cdot}| < d \leq |E_{k_0, \cdot}|$$

Hence for $k \leq k_0$ we have $r_{\tau_k} \leq d + |E_{\tau_k}| \leq 2|E_{\tau_k}|$ and so

$$\begin{aligned} \Re(F(z)) &= \sum_{\tau \in \mathcal{I}} a_k \cos(\alpha_k \theta_\tau) r_\tau^{-\alpha_k} \geq \sum_{k=1}^{k_0} a_k \cos(\alpha_k \frac{\pi}{2}) r_{\tau_k}^{-\alpha_k} \\ &\geq \frac{1}{2} \sum_{k=1}^{k_0} a_k \cos(\alpha_k \frac{\pi}{2}) |E_{\tau_k}|^{-\alpha_k}. \end{aligned}$$

We will consider $0 < s < 1$ and $s = 1$ separately.

If $0 < s < 1$ we have $a_k \cos(\alpha_k \frac{\pi}{2}) |E_{\tau_k}|^{-\alpha_k} = k^{-2} \cos(\alpha \frac{\pi}{2}) \zeta^k$ where $\zeta = 2^{\frac{\alpha}{s}-1} > 1$.

We combine this with

$$(\zeta - 1) \sum_{k=1}^{k_0} k^{-2} \zeta^k = k_0^{-2} \zeta^{k_0+1} - \zeta + \sum_{k=1}^{k_0-1} (k^{-2} - (k+1)^{-2}) \zeta^{k+1} \geq k_0^{-2} \zeta^{k_0+1} - \zeta$$

to conclude that

$$\begin{aligned} \Re(F(z)) + m \ln(d) &\geq c k_0^{-2} \zeta^{k_0+1} + m \ln(d) - c \zeta \\ &\geq c k_0^{-2} \zeta^{k_0+1} - \frac{m \ln(2)}{s} (k_0 + 1) - c \zeta \rightarrow +\infty \quad (k_0 \rightarrow \infty) \end{aligned}$$

where $c = \frac{\cos(\alpha \frac{\pi}{2})}{2(\zeta-1)}$. This is equivalent to (2.6.8) since due to (2.6.9), $-\frac{\ln(2)}{s} (k_0 + 1) <$

$\ln(d) \leq -\frac{\ln(2)}{s} k_0$.

If $s = 1$, we have

$$(2.6.10) \quad a_k \cos(\alpha_k \frac{\pi}{2}) |E_{\tau_k}|^{-\alpha_k} \geq \frac{1}{2} \frac{2^{\frac{1}{4}k^{\frac{2}{3}}}}{k^{\frac{7}{3}}} \text{ for } k \geq 9.$$

(2.6.10) holds because firstly $|E_{\tau_k}| = 2^{-k-k^{\frac{2}{3}}}$, $\alpha_k = 1 - \frac{1}{2}k^{-\frac{1}{3}}$ and therefore

$$\frac{\ln(2^k |E_{\tau_k}|^{-\alpha_k})}{\ln(2)} = (1 - \frac{1}{2}k^{-\frac{1}{3}})(k + k^{\frac{2}{3}}) - k \geq \frac{k^{\frac{2}{3}}}{4} \text{ for } k \geq 9.$$

Secondly, $\cos(\alpha_k \frac{\pi}{2}) \geq (1 - \alpha_k) = \frac{k^{-\frac{1}{3}}}{2}$ because $\cos((1-t)\frac{\pi}{2}) \geq t$ for $0 \leq t \leq 1$. Similar as before we have

$$(2.6.11) \quad (2^{\frac{1}{6}} - 1) \sum_{k=9}^{k_0} \frac{2^{\frac{k^{\frac{2}{3}}}{4}}}{k^{\frac{7}{3}}} = \frac{2^{\frac{k_0^{\frac{2}{3}}}{4} + \frac{2}{3}}}{k_0^{\frac{7}{3}}} - \frac{2^{\frac{9^{\frac{2}{3}}}{4}}}{9^{\frac{7}{3}}} + \sum_{k=9}^{k_0-1} \frac{2^{\frac{k^{\frac{2}{3}}}{4} + \frac{2}{3}}}{k^{\frac{7}{3}}} - \frac{2^{\frac{(k+1)^{\frac{2}{3}}}{4}}}{(k+1)^{\frac{7}{3}}} \geq \frac{2^{\frac{(k_0+1)^{\frac{2}{3}}}{4}}}{k_0^{\frac{7}{3}}} - 1,$$

where we used that $k^{\frac{2}{3}} + \frac{2}{3} \geq (k+1)^{\frac{2}{3}}$ to conclude that the sum in the middle is non-negative. We combine (2.6.10) and (2.6.11) to conclude

$$\begin{aligned} \Re(F(z)) + m \ln(d) &\geq \sum_{k=9}^{k_0} a_k \cos(\alpha_k \frac{\pi}{2}) |E_{\tau_k}|^{-\alpha_k} + m \ln(d) \\ &\geq c \frac{2^{\frac{(k_0+1)^{\frac{2}{3}}}{4}}}{k_0^{\frac{7}{3}}} - c - m \ln(2)(k_0 + 1 + (k_0 + 1)^{\frac{2}{3}}) \rightarrow +\infty \quad (k_0 \rightarrow \infty) \end{aligned}$$

where $c = \frac{1}{4(2^{\frac{1}{6}} - 1)}$. As before it is equivalent to (2.6.8) because of (2.6.9), which is

equivalent to $-\ln(2)(k_0 + 1 + (k_0 + 1)^{\frac{2}{3}}) < \ln(d) \leq -\ln(2)(k_0 + k_0^{\frac{2}{3}})$. \square

2.7. APPLICATIONS

2.7.1. Minimal surfaces. Given a holomorphic function h on $\Omega \subset \mathbb{C}$ open, $Q \in \mathbb{N}$ one defines the irreducible holomorphic variety $\mathcal{V} \subset \Omega \times \mathbb{C}$ by

$$(2.7.1) \quad \mathcal{V} = \{(z, u) \in \Omega \times \mathbb{C} : u^Q = h(z)\}.$$

Following Federer we associate to \mathcal{V} an integer rectifiable current of real dimension two denoted by $[\mathcal{V}]$. It is given by integration over the manifold part of \mathcal{V} , $\mathcal{V}_{reg.}$ i.e. $\mathcal{V}_{reg.} = \{(z, u) : u^Q = h(z), h(z) \neq 0\}$.

Federer observed that $[\mathcal{V}]$ is a mass-minimizing cycle, since \mathcal{V} , as a complex submanifold of \mathbb{C}^2 is calibrated by the Kähler form (Wirtinger's form).

If we take $h = g$, $\Omega = \mathbb{C}_+$ in (2.7.1) we get the following example:

Example 2.7.1. Given $0 < s \leq 1$ and an integer $Q \geq 2$ there is a mass-minimizing cycle $\mathcal{V} \subset \mathbb{C}_+ \times \mathbb{C}$ with the additional property that if $s < 1$ then $\mathcal{H}^s(\overline{\mathcal{V} \setminus \mathcal{V}_{reg.}}) = 1$ and if $s = 1$ then $\dim_{\mathcal{H}}(\overline{\mathcal{V} \setminus \mathcal{V}_{reg.}}) = 1$.

The additional property holds since $\mathcal{V} \setminus \mathcal{V}_{reg.} = \{(z, 0) \in \mathbb{C}_+ \times \mathbb{C} : G(z) = 0\}$ and therefore $\overline{\mathcal{V} \setminus \mathcal{V}_{reg.}} = \{(z, 0) \in \mathbb{C}_+ \times \mathbb{C} : G(z) = 0\} \cup -iE_s$. $\{(z, 0) \in \mathbb{C}_+ \times \mathbb{C} : G(z) = 0\}$ is countable so that the claim follows by the properties of E_s .

Remark 2.7.2. For two dimensional minimal surfaces in \mathbb{R}^3 R. Ossermann had shown in [18] that true branching points can be ruled out in the interior. If the boundary curve is real analytic the existence branching points at the boundary can be ruled out as well. This was shown by R. Gulliver and F. Leslie in [9] for two dimensional surfaces in \mathbb{R}^3 .

R. Gulliver presents in [8, Theorem 1.6] the following example:

Theorem 2.7.1. *There is a smooth minimal immersion $X(\Omega) \subset \mathbb{R}^3$, $\Omega \subset \mathbb{C}_+$ simply connected with the following property: X maps $\partial\Omega$ diffeomorphically onto a regular C^∞ Jordan curve $\Gamma \subset \mathbb{R}^3$ and has a true branch point at $z = 0 \in \Gamma$. The set of self intersections of X consists of the union of an infinite sequence of disjoint real analytic arcs, each which joints two points of Γ lying on opposite sides of the branch point.*

His construction uses the Weierstrass representation with a holomorphic vector-field that comes from a perturbation of the building block $a(z) = e^{-z^\alpha}$, (2.4.1), with $\alpha = \frac{1}{7}$. It could be of interest to see if one can follow his analysis using one of the holomorphic functions f or g (lemma 2.4.1) to construct a minimal immersion X in \mathbb{R}^3 with C^∞ boundary curve and a large set of true branching points on the boundary.

2.7.2. Dirichlet minimizing Q -valued functions. One of the implications of lemma 2.4.1 in the context of Q -valued functions had been stated heuristically in the introduction. The holomorphic functions f, g generate examples of Q -valued functions that are Dirichlet minimizing with respect to compact perturbations. Furthermore these examples are defined on $\mathbb{R}_+^2 = \{(x, y) \in \mathbb{R}^2 : x > 0\} \simeq \mathbb{C}_+$ and have "large" singular set towards the boundary. As we mentioned before the classical theory of Dirichlet minimizing Q -valued functions had been developed in [1] and revisited with modern methods in [12].

Before we are going to state the precise properties of the examples we recall the definition of the singular set and related results thereafter the definition of $C^k(\Omega, \mathcal{A}_Q(\mathbb{R}^m))$ for a domain $\Omega \subset \mathbb{R}^n$.

Definition of the singular set:

Given a Dirichlet minimizer $u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^m))$, $\Omega \subset \mathbb{R}^N$ open, a point $y \in \Omega$

is called a regular point of u if $\exists U \subset \Omega$ open neighborhood of y , $u_i \in C^\infty(U, \mathbb{R}^m)$ harmonic with

$$u(x) = \sum_{i=1}^Q \llbracket u_i(x) \rrbracket \text{ for a.e. } x \in U$$

and $u_i(x) \neq u_j(x), \forall x \in U$ or $u_i \equiv u_j$. The open set (by definition) of all regular points is denoted by $\text{reg}(u)$. $\text{sing}(u)$ then denotes the relative closed complement $\Omega \setminus \text{reg}(u)$.

An outcome of Almgrens original work is an estimate on the size of the singular set in the interior, compare [12, Theorem 0.11].

Theorem 2.7.2. $u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^m))$ Dirichlet minimizing has $\dim_{\mathcal{H}}(\text{sing}(u)) \leq N - 2$. In the case of $N = 2$, $\text{sing}(u)$ is countable.

This estimate had been improved by C. De Lellis and E. Spadaro, [12, Theorem 0.12].

Theorem 2.7.3. u as above and $N = 2$ then $\text{sing}(u)$ consists of isolated points.

That the upper bound on the Hausdorff dimension is sharp is a consequence of the following:

Theorem 2.7.4. Let $\mathcal{V} \subset \mathbb{C}^N \times \mathbb{C}^m \simeq \mathbb{R}^{2N} \times \mathbb{R}^{2m}$ be an irreducible holomorphic variety with the property that $\exists \Omega \subset \mathbb{C}^N$ open, C^1 -regular, \mathcal{V} is a $Q : 1$ cover of Ω under the orthogonal projection and $\mathbf{M}(\mathcal{V} \cap (\Omega \times \mathbb{C}^m)) < \infty$. Then $\exists u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^{2m}))$ Dirichlet minimizing with $\text{graph}(u) = \mathcal{V} \cap (\Omega \times \mathbb{C}^m)$.

This was originally proven by Almgren, [1, Theorem 2.20]. E. Spadaro found a very elegant more elementary proof, [22, Theorem 0.1].

Hence the holomorphic varieties $\mathcal{V} = \mathcal{V}_h$ defined in (2.7.1) generate examples of Dirichlet minimizers:

$$(2.7.2) \quad u_h(z) = \sum_{\substack{v \in \mathbb{C} \\ v^Q = h(z)}} \llbracket v \rrbracket \text{ for } z \in \Omega.$$

Definition of $C^k(\Omega, \mathcal{A}_Q(\mathbb{R}^m))$:

Let Ω be a domain in \mathbb{R}^n and let $\bigwedge^m(\mathbb{R}^N, \mathbb{R}^m)$ denote the space of m -linear maps from \mathbb{R}^N to \mathbb{R}^m . $u \in C^0(\Omega, \mathcal{A}_Q(\mathbb{R}^m))$ is said to be in $C^k(\Omega, \mathcal{A}_Q(\mathbb{R}^m))$ if: $\exists \mathcal{U}^k : \Omega \rightarrow \mathcal{A}_Q(\mathbb{R}^m \times \bigwedge^1(\mathbb{R}^N, \mathbb{R}^m) \times \dots \times \bigwedge^k(\mathbb{R}^N, \mathbb{R}^m))$ continuous,

$$x \mapsto \mathcal{U}_x = \sum_{i=1}^Q \llbracket (u_i(x), U_i^1(x), \dots, U_i^k(x)) \rrbracket$$

with $U_i^m(x) = U_j^m(x)$ whenever $u_i(x) = u_j(x)$. \mathcal{U}^k defines the k -jet $J\mathcal{U}^k : \Omega \times \mathbb{R}^N \rightarrow \mathcal{A}_Q(\mathbb{R}^m)$ by

$$J\mathcal{U}_x^k(y) = \sum_{i=1}^Q \llbracket u_i(x) + U^1(x)(y-x) + \dots + \frac{1}{k!} U^k(x)((y-x) \wedge \dots \wedge (y-x)) \rrbracket$$

and it satisfies

$$\mathcal{G}(u(y), J\mathcal{U}_x^k(y)) = o(|x-y|^k) \quad \forall x \in \Omega.$$

It is straightforward to check that if $u_i \in C^k(\Omega, \mathbb{R}^m)$ with $u_i(x) \neq u_j(x) \forall x \in \Omega$ or

$$u_i \equiv u_j,$$

$$(2.7.3) \quad u(x) = \sum_{i=1}^Q \llbracket u_i(x) \rrbracket$$

is in $C^k(\Omega, \mathcal{A}_Q(\mathbb{R}^m))$ with $\mathcal{U}^k = \sum_{i=1}^Q \llbracket (u_i(x), Du_i(x), \dots, D^k u_i(x)) \rrbracket$.

Now we are able to state properly the properties of the examples:

Corollary 2.7.5. *Let $0 < s \leq 1$ and an integer $Q \geq 2$ be given, then there is $u \in W_{loc.}^{1,2}(\mathbb{R}_+, \mathcal{A}_Q(\mathbb{R}^2))$, Dirichlet minimizing with respect to compact perturbations of \mathbb{R}^2 and the additional properties*

(i) $u|_{\partial \mathbb{R}_+^2} \in C^k(\partial \mathbb{R}_+^2, \mathcal{A}_Q(\mathbb{R}^2))$ for all $k \in \mathbb{N}$;

(ii) if $s < 1$ then $\mathcal{H}^s(\overline{\text{sing}(u)}) = 1$ and if $s = 1$ then $\dim_{\mathcal{H}}(\overline{\text{sing}(u)}) = 1$.

Proof of lemma 2.7.5. Let $0 < s \leq 1$ be fixed and $g(z) = G(z)e^{-F(z)}$ be the holomorphic function on \mathbb{C}_+ constructed in lemma 2.4.1.

$$u(z) = \sum_{\substack{v \in \mathbb{C} \\ v^Q = g(z)}} \llbracket v \rrbracket \quad z \in \mathbb{C}_+$$

is Dirichlet minimizing and an element of $W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^2))$ for any C^1 -regular bounded subset $\Omega \subset \mathbb{C}_+$ as a consequence of theorem 2.7.4.

It remains to check the C^∞ -regularity at the boundary and the property of the singular set.

We start with the regularity of the trace. By construction we had $g(z) = G(z)e^{-F(z)}$ is holomorphic on $\mathbb{C} \setminus (\mathbb{R}_- - iE_s)$ and $g|_{\mathbb{C}_+}$ has an C^∞ extension to \mathbb{C}^+ . Furthermore $G(z) \neq 0$ for all $z \in \mathbb{C} \setminus (\mathbb{R}_- - iE_s)$, $|G(z)| < C$ uniformly on $\mathbb{C} \setminus (\mathbb{R}_- - iE_s)$. So that for any $B_r(z_0) \subset \mathbb{C} \setminus (\mathbb{R}_- - iE_s)$ there exists a holomorphic branch $\psi : G(B_r(z_0)) \rightarrow \mathbb{C}$ of the Q -th. root. u is then explicitly given by

$$u(z) = \sum_{l=0}^{Q-1} \llbracket \xi^l (\psi \circ G)(z) e^{-\frac{1}{Q}F(z)} \rrbracket \quad \forall z \in B_r(z_0), \xi = e^{i\frac{2\pi}{Q}}.$$

Hence we are in the situation of (2.7.3) on $B_r(z_0)$. The k -jet of u is

$$\mathcal{U}_z^k = \sum_{l=0}^Q \llbracket (\xi^l (\psi \circ g)(z), \xi^l (\psi \circ g)^{(1)}(z), \dots, \xi^l (\psi \circ g)^{(k)}(z)) \rrbracket$$

where we write $\psi \circ g(z)$ for $(\psi \circ G)(z) e^{-\frac{1}{Q}F(z)}$. The C^∞ -regularity will follow from

$$(2.7.4) \quad |(\psi \circ g)^{(m)}(-iy)| = O(\text{dist}(y, E_s)) \quad \text{for all } m \in \mathbb{N}.$$

The same arguments used in the proof to lemma 2.6.4 show that

$$|F^{(m)}(z)e^{-\frac{1}{Q}F(z)}| \leq C d^{-m-1} |e^{-\frac{1}{Q}F(z)}| = C \left(d^{-Q(m+1)} e^{-\Re(F(z))} \right)^{\frac{1}{Q}}$$

for all $z \in \{\text{dist}(z, -iE_s) \geq d\}$ and a constant $C = C(m) > 0$. Let $z \in \{\text{dist}(z, \mathbb{R} - iE_s) > d\}$ be given, then $\psi \circ G$ is holomorphic on $B_d(z)$. So Cauchy's integral formula gives

$$(\psi \circ G)^{(m)} = \frac{m!}{2\pi i} \oint_{\partial B_d(z)} \frac{\psi \circ G(w)}{(w-z)^{m+1}} dw$$

and therefore

$$|(\psi \circ G)^{(m)}(z)| \leq \frac{m!}{d^m} \sup_{w \in B_d(z)} |G(w)|^{\frac{1}{Q}} \leq C m! d^{-m}.$$

We used the uniform bound on $|G|$. Hence we deduce

$$|(\psi \circ G)^{(m)}(z)e^{-\frac{1}{Q}F(z)}| \leq C \left(d^{-Qm} e^{-\Re(F(z))} \right)^{\frac{1}{Q}} \quad \forall z \in \{\text{dist}(z, \mathbb{R} - iE_s) > d\}.$$

So (2.7.4) follows from (2.6.8) where we showed that for any $m \in \mathbb{N}$

$$\Re(F(z)) + m \ln(d) \rightarrow +\infty \text{ as } d \rightarrow 0.$$

It remains to check the properties of the singular set. By construction of u we have

$$\overline{\text{sing}(u)} = \{z \in \mathbb{C}_+ : g(z) = 0\} \cup -iE_s$$

because g has the property that to any $z \in -iE_s$ there exists $z_k \in \mathbb{C}_+$, $z_k \rightarrow 0$ and $g(z_k) = 0$. Set $A_k = \{z \in \mathbb{C}_+ : g(z) = 0, 2^k \leq \Re(z) < 2^{k+1}\}$ for any $k \in \mathbb{Z}$. A_k consists of isolated points since g is holomorphic on \mathbb{C}_+ and therefore $\mathcal{H}^s(A_k) = 0$ for all $k \in \mathbb{Z}$ and $s > 0$. Hence we deduce

$$\mathcal{H}^s(-iE_s) \leq \mathcal{H}^s(\overline{\text{sing}(u)}) \leq \mathcal{H}^s(-iE_s) + \sum_{k \in \mathbb{Z}} \mathcal{H}^s(A_k) = \mathcal{H}^s(-iE_s).$$

□

This example, corollary 2.7.5, shows that the singular set can behave very badly towards the boundary. In the interior a blow-up analysis together with a Federer reduction argument is used to study the singular set, compare [12, section 3]. With the following calculation we want to show that this procedure cannot directly be transferred to the boundary.

Almgren's celebrated frequency function is the major tool to carry out the blow-up analysis. For $u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^m))$ with $\Omega \subset \mathbb{R}^N$ open it is defined as

$$(2.7.5) \quad I(u, y, r) = \frac{D(u, y, r)}{H(u, y, r)} = \frac{r^{2-N} \int_{B_r(y) \cap \Omega} |Du|^2}{r^{1-N} \int_{\partial B_r(y)} |u|^2}.$$

Its essential property is, compare [12, Theorem 3.15]

Theorem 2.7.6. *Let $u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^m))$ be Dirichlet minimizing, then for any $y \in \Omega$ either $\exists 0 < R < \text{dist}(y, \partial\Omega)$ s.t. $u|_{B_R(y)} \equiv 0$ or $r \mapsto I(u, y, r)$ is absolutely continuous, nondecreasing and positive.*

Consequently the following limit is well-defined in the interior of Ω

$$(2.7.6) \quad I(u, y) = \lim_{r \rightarrow 0} I(u, y, r)$$

In the planar case C. De Lellis and E. Spadaro determined the spectrum of $y \mapsto I(u, y)$ to be $\{\frac{P}{Q} : P \in \mathbb{N}\} \cup \{0\}$, [12, Proposition 5.1].

The following examples show that this may fail at boundary points.

Corollary 2.7.7. *Let $Q \geq 2, P > 0$ be two divisor free integers then there exists a Dirichlet minimizer $u \in W_{loc}^{1,2}(\mathbb{R}_+^2, \mathcal{A}_Q(\mathbb{R}^2))$ with*

- (i) $u|_{\partial\mathbb{R}_+^2} \in C^k(\partial\mathbb{R}_+^2, \mathcal{A}_Q(\mathbb{R}^2));$
- (ii) for all $k \in \mathbb{N}$, $z_k = (e^{-k\pi + \frac{\pi}{2}}, 0)$ is a branch point of "order" $\frac{P}{Q}$ i.e.
 $I(u, z_k) = \frac{P}{Q};$
- (iii) $\lim_{r \rightarrow 0} I(u, 0, r) = +\infty.$

Corollary 2.7.8. *Let $Q > 2$ be an integer, $0 < s < 1$ be given there is a Dirichlet minimizer $u \in W_{loc}^{1,2}(\mathbb{R}_+^2, \mathcal{A}_Q(\mathbb{R}^2))$ with*

- (i) $u|_{\partial\mathbb{R}_+^2} \in C^k(\partial\mathbb{R}_+^2, \mathcal{A}_Q(\mathbb{R}^2));$
- (ii) $\text{sing}(u) = \emptyset$, but $u(z) = Q\llbracket 0 \rrbracket \quad \forall z \in -iE_s$ with $\mathcal{H}^s(E_s) = 1$;

- (iii) $\lim_{n \rightarrow \infty} I(u, -iy_k, R_n) = +\infty$ for a countable subset $\{y_k\}_{k \in \mathbb{N}} \subset E_s$ and a sequence $R_n \rightarrow 0$.

Before we give the proofs, we collect two observations to calculate energy and L^2 -norm for multivalued functions arising from the holomorphic varieties defined in (2.7.4).

$\mathcal{A}_Q(\mathbb{C}) \simeq \mathcal{A}_Q(\mathbb{R}^2)$ enables us to define a Q -root "globally", i.e. an "inverse" to the holomorphic function $z \mapsto z^Q$ by

$$(2.7.7) \quad \Pi(w) = \sum_{v^Q=w} \llbracket v \rrbracket = \sum_{l=0}^{Q-1} \llbracket \xi^l v_0 \rrbracket$$

for $\xi = e^{i\frac{2\pi}{Q}}$ and an arbitrary choice of $v_0 \in \mathbb{C}$ with $v_0^Q = w$. Furthermore we observed already before that for $y \in \Omega$ with $h(y) \neq 0$ there is an open neighborhood U with $|h(z) - h(y)| < |h(y)|, \forall z \in U$. There is an holomorphic branch ψ of the Q -root on $|w - h(y)| < |h(y)|$ so that $\Pi(w) = \sum_{l=0}^{Q-1} \llbracket \xi^l \psi(w) \rrbracket$ on $B_{|h(y)|}(h(y))$ showing that Π is continuous on all of \mathbb{C} . Furthermore

$$(2.7.8) \quad u(z) = \Pi \circ h(z) = \sum_{l=0}^{Q-1} \llbracket \xi^l (\psi \circ h)(z) \rrbracket \quad \forall z \in U.$$

Hence $u \in C^k(U, \mathcal{A}_Q(\mathbb{R}^2))$ for all k since we are in the situation mentioned in (2.7.3) with

$$(2.7.9) \quad \mathcal{U}_z^k = \sum_{l=0}^{Q-1} \llbracket (\xi^l (\psi \circ h)(z), \xi^l (\psi \circ h)^{(1)}(z), \dots, \xi^l (\psi \circ h)^{(k)}(z)) \rrbracket \quad \forall z \in U.$$

We note that \mathcal{U}^k does not depend on the particular choice of the branch.

As an immediate consequence of (2.7.8) the L^2 norm of u is given by

$$(2.7.10) \quad \int_{V \cap \Omega} |u|^2 = Q \int_{V \cap \Omega} |h|^{\frac{2}{Q}}$$

for any $V \subset \mathbb{C}$. The energy of u on $V \cap \Omega$ due to (2.7.9) is then

$$(2.7.11) \quad \int_{V \cap \Omega} |Du|^2 = 2Q \int_{V \cap \Omega \setminus \{h \neq 0\}} |(\psi \circ h)'|^2 = \frac{2}{Q} \int_{V \cap \Omega \setminus \{h \neq 0\}} |h|^{\frac{2}{Q}-2} |h'|^2$$

where ψ is any local choice of a branch ψ to the Q -root.

For instance we can use it to calculate the value of the frequency at interior branch points.

Example 2.7.3. Let h be holomorphic on $\Omega \subset \mathbb{C}$ and u the related Dirichlet minimizer (see (2.7.2)). Let $z_0 \in \Omega$ be a zero of order $P \geq 1$ then

$$I(u, z_0) = \frac{P}{Q}.$$

z_0 is a zero of order P , hence there is k holomorphic on $\{z: |z| < \delta\}$, $k_0 = k(0) \neq 0$ s.t. $h(z_0 + z) = z^P k(z)$. We may assume that $|k(z)| > \frac{1}{2}|k_0|^2$ for all $|z| < \delta$. $h'(z_0 + z) = Pz^{P-1}k(z)(1 + \frac{zk'(z)}{Pk(z)}) = \frac{P}{z}h(z_0 + z)(1 + o(z))$ and so we may use $|h|^{\frac{1}{Q}-1}|h'| (z_0 + z) = P|z|^{\frac{P}{Q}-1}|k_0|^{\frac{1}{Q}}(1 + o(z))$ in (2.7.11) to deduce

$$\int_{B_r(z_0)} |Du|^2 = \frac{2P^2}{Q} \int_{B_r(0)} |z|^{\frac{2P}{Q}-2} |k_0|^{\frac{2}{Q}} (1 + o(z)) = 2\pi P |k_0|^{\frac{2}{Q}} r^{\frac{2P}{Q}} (1 + o(r))$$

for any $0 < r < \delta$. Similarly, using (2.7.10) we have

$$\frac{1}{r} \int_{\partial B_r(z_0)} |u|^2 = \frac{Q}{r} \int_{\partial B_r} |z|^{\frac{2P}{Q}} |k_0|^{\frac{2}{Q}} (1 + o(z)) = 2\pi Q |k_0|^{\frac{2}{Q}} r^{\frac{2P}{Q}} (1 + o(r)).$$

We conclude the claim:

$$I(u, z_0, r) = \frac{P}{Q}(1 + o(r)).$$

For boundary points $z_0 \in \partial\Omega$ we are facing two problems to estimate $I(u, z_0, r)$ and possible limits. Firstly $r \mapsto I(u, z_0, r)$ is a priori not a monotone quantity as it is in the interior. Secondly, even restricting ourselves to minimizers of the type (2.7.2), $h(z)$ does not necessarily have a convergent Taylor series at z_0 . The strategy will be to use the mean value theorem for integration in the radial variable to estimate $D(u, z_0, r) = \int_{B_r(z_0) \cap \Omega} |Du|^2$ from below by a multiple of $H(u, z_0, r) = \frac{1}{r} \int_{\partial B_r(z_0) \cap \Omega} |u|^2$. The strategy is motivated by the following observation. Given a function k holomorphic in a neighbourhood of $z \in \mathbb{C}$ and $k(z) \neq 0$, $\gamma > 0$, for any $\xi = e^{i\theta}$ one has

$$D_\xi |k|^2 = 2\Re(\bar{k}k'\xi) = 2|k|^2 \Re\left(\frac{k'}{k}\xi\right)$$

and so $D_\xi |k|^\gamma = \frac{\gamma}{2}|k|^{\gamma-2} D_\xi |k|^2 = \gamma|k|^{\gamma-2} \Re\left(\frac{k'}{k}\xi\right)$. This gives

$$(2.7.12) \quad \gamma|k|^{\gamma-2}|k'|^2 = \gamma|k|^\gamma \left|\frac{k'}{k}\right|^2 \geq \gamma|k|^\gamma \Re\left(\frac{k'}{k}\xi\right)^2 = \Re\left(\frac{k'}{k}\xi\right) D_\xi |k|^\gamma.$$

The strategy is illustrated in the following example:

Example 2.7.4. Let $h(z) = e^{-z^{-\alpha}}$, $0 < \alpha < 1$ ($h(z) = a(z)$ of (2.4.1)) in (2.7.2), i.e. $u(z) = \sum_{v \in \mathbb{C}} \llbracket v \rrbracket$ with $z \in \Omega = \mathbb{C}_+$, then u satisfies

$$\lim_{R \rightarrow 0} I(u, 0, R) = +\infty.$$

We will use the classic radial notation $z = re^{i\theta}$. We define

$$\varphi(z) = r \Re\left(\frac{h'(z)}{h(z)} e^{i\theta}\right) = \alpha \Re(z^{-\alpha}) = \alpha r^{-\alpha} \cos(\alpha\theta).$$

Combining (2.7.11) with (2.7.12) ($h(z) \neq 0 \forall z \in \mathbb{C}_+$) gives

$$\begin{aligned} \int_{B_R \cap \mathbb{C}_+} |Du|^2 &= \int_{B_R \cap \mathbb{C}_+} \frac{2}{Q} |h(z)|^{\frac{2}{Q}-2} |h'|^2 \geq \int_{B_R \cap \mathbb{C}_+} \frac{\varphi(z)}{r} \frac{\partial}{\partial r} |h|^{\frac{2}{Q}} \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^R \varphi(re^{i\theta}) \left(\frac{\partial}{\partial r} |h|^{\frac{2}{Q}}\right) (re^{i\theta}) dr d\theta \end{aligned}$$

Since $\varphi(z) \geq \alpha r^{-\alpha} \cos(\alpha\frac{\pi}{2}) > 0$, (2.7.12) implies that $\frac{\partial}{\partial r} |h|^{\frac{2}{Q}} \geq 0$. Thus we apply the 1-dimensional mean value theorem to deduce that to every $|\theta| \leq \frac{\pi}{2}$ there is $0 < r_\theta \leq R$ with

$$\begin{aligned} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^R \varphi(re^{i\theta}) \left(\frac{\partial}{\partial r} |h|^{\frac{2}{Q}}\right) (re^{i\theta}) dr d\theta &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \varphi(r_\theta e^{i\theta}) \int_0^R \left(\frac{\partial}{\partial r} |h|^{\frac{2}{Q}}\right) (re^{i\theta}) dr d\theta \\ &\geq \alpha R^{-\alpha} \cos(\alpha\frac{\pi}{2}) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |h|^{\frac{2}{Q}} (Re^{i\theta}) d\theta. \end{aligned}$$

(Although it is not needed for the argument that the map $\theta \mapsto \varphi(r_\theta e^{i\theta})$ is measurable, since it is sufficient that it is point wise bounded, we included a short remark below on the measurability.) We conclude using (2.7.10) that

$$\int_{B_R \cap \mathbb{C}_+} |Du|^2 \geq \frac{\alpha}{Q} R^{-\alpha} \cos(\alpha\frac{\pi}{2}) \frac{1}{R} \int_{\partial B_R \cap \mathbb{C}_+} |u|^2$$

i.e. $I(u, 0, R) \geq \frac{\alpha}{Q} R^{-\alpha} \cos(\alpha\frac{\pi}{2}) \rightarrow +\infty (R \rightarrow 0)$.

As we mentioned in the proof we give a short comment concerning the measurability.

Remark 2.7.5. We will prove the following claim:

Let μ be a Borel regular measure on a path-connected space X , ν a measure on some space Y and $\mu \times \nu$ the product measure on $X \times Y$. Given f, g with the properties that

- (i) f, g, fg are $\mu \times \nu$ summable, i.e. $f, g, fg \in L^1(X \times Y, \mu \times \nu)$;
- (ii) $x \mapsto f(x, y)$ is continuous for a.e. y .

Then there exists a map $\chi : Y \rightarrow X$ s.t.

$$(2.7.13) \quad y \mapsto f(\chi(y), y) \int_X g(x, y) d\mu(x) = \int_X fg(x, y) d\mu(x) \text{ is } \nu\text{-integrable and}$$

$$(2.7.14) \quad f(\chi(y), y) \int_X g(x, y) d\mu(x) = \int_X fg(x, y) d\mu(x) \text{ for a.e. } y$$

Indeed, let $A \subset Y$ be the set of $y \in Y$ s.t.

- (a) $x \mapsto f(x, y)$ is continuous and $|f|$ is finite;
- (b) $x \mapsto g(x, y), fg(x, y)$ are μ -summable ($g(\cdot, y), fg(\cdot, y) \in L^1(X, \mu)$).

We have $\nu(Y \setminus A) = 0$ since (a) holds for a.e. y by assumption and (b) holds for a.e. y by general measure theory. The 1-dimensional mean value theorem tells that for $y \in A$ there exists $\chi(y) \in X$ s.t. the identity (2.7.14) holds. Indeed let $y \in A$ be fixed, then $z \mapsto f(z, y) \int_X g(x, y) d\mu(x)$ is continuous and since $|\int_X f(x, y)g(x, y) d\mu(x)| < \infty$ we can find $x_0, x_1 \in X$ s.t.

$$\begin{aligned} \inf_{z \in X} f(z, y) \int_X g(x, y) d\mu(x) &\leq f(x_0, y) \int_X g(x, y) d\mu(x) \\ &\leq \int_X f(x, y)g(x, y) d\mu(x) \\ &\leq f(x_1, y) \int_X g(x, y) d\mu(x) \leq \sup_{z \in X} f(z, y) \int_X g(x, y) d\mu(x). \end{aligned}$$

By assumption there is a continuous path γ connecting x_0 with x_1 . Now we may apply the 1-dimensional mean value theorem to $t \mapsto f(\gamma(t), y) \int_X g(x, y) d\mu(x)$ to find a point $\chi(y)$. Since $\int_X (fg)(x, y) d\mu(x)$ is ν -integrable and for all $y \in A$ (2.7.14) is satisfied (2.7.13) holds. If in addition $\int_X g(x, y) d\mu(x) \neq 0$ for a.e. y then $y \mapsto f(\chi(y), y)$ is ν -measurable.

Proof of corollary 2.7.7. We claim that the minimizer $u(z) = \sum_{\substack{v \in \mathbb{C} \\ v^Q = b^P(z)}} \llbracket v \rrbracket$ with

$b(z) = \cos(\ln(z))e^{-z^{-\alpha}}$ (compare (2.4.1)) has the desired properties.

(i) follows from the same arguments presented in the proof of corollary 2.7.5 so we omit the details here.

(ii) corresponds to example 2.7.3. Since $\{z \in \mathbb{C}_+ : b(z) = 0\} = \{e^{\frac{\pi(2k+1)}{2}} : k \in \mathbb{Z}\}$, $b'(e^{\frac{\pi(2k+1)}{2}}) = (-1)^{k+1}e^{-e^{-\alpha} \frac{\pi+2k}{2}} \neq 0$ and so $e^{-\frac{\pi(2k+1)}{2}}$ is a zero of order P to $b(z)^P$.

(iii) remains to be proven. We want to do it similarly to the example 2.7.4. As before we define

$$\varphi(z) = \Re \left(\frac{b'(z)}{b(z)} z \right) = \Re \left(\alpha z^{-\alpha} - \frac{\sin(\ln(z))}{\cos(\ln(z))} \right).$$

$\Re(\tan(\ln(re^{i\theta})))$ is not uniformly bounded as $|\theta| \rightarrow 0$, hence we can not conclude directly $\varphi(re^{i\theta}) \geq 0$ for $r > 0$ sufficient small. But $|\tan(\ln(re^{i\theta}))|^2 \leq \frac{1}{\tanh(\theta)^2}$ is

bounded on $\frac{\pi}{4} \leq |\theta| \leq \frac{\pi}{2}$ and so

$$(2.7.15) \quad \varphi(re^{i\theta}) \geq \alpha r^{-\alpha} \cos(\alpha \frac{\pi}{2}) - \frac{1}{\tanh(\frac{\pi}{4})} \geq 0$$

for $\frac{\pi}{4} \leq |\theta| \leq \frac{\pi}{2}$ and $0 < r \leq R$, $R > 0$ sufficient small.

$$\lambda \mapsto |b(re^{i\lambda\theta})|^2 = |\cos(\ln(re^{i\lambda\theta}))|^2 e^{-2r^{-\alpha} \cos(\alpha\lambda\theta)}$$

is monotone increasing. $\lambda \mapsto e^{-r^{-\alpha} \cos(\alpha\lambda\theta)}$ is monotone increasing for $|\lambda\alpha\theta| \leq \frac{\pi}{2}$ and $\lambda \mapsto |\cos(\ln(re^{i\lambda\theta}))|^2$ because $\frac{\partial}{\partial \lambda} |\cos(\ln(re^{i\lambda\theta}))|^2 = \sinh(2\lambda\theta)\theta \geq 0$. Combine it with (2.7.10) ($|h|^2 = |b|^{2P}Q$) give

$$(2.7.16) \quad \begin{aligned} \frac{1}{R} \int_{B_R \cap \mathbb{C}_+} |u|^2 &= Q \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |b(Re^{i\theta})|^{\frac{2Q}{P}} d\theta \leq Q \int_{\frac{\pi}{4} < |\theta| < \frac{\pi}{2}} |b(Re^{i\theta})|^{\frac{2Q}{P}} d\theta \\ &+ Q \int_{|\theta| < \frac{\pi}{4}} |b(Re^{i2\theta})|^{\frac{2Q}{P}} d\theta = \frac{3Q}{2} \int_{\frac{\pi}{4} < |\theta| < \frac{\pi}{2}} |b(Re^{i\theta})|^{\frac{2Q}{P}} d\theta. \end{aligned}$$

(2.7.11) together with (2.7.12) gives with $h = b^P$, $h' = Pb^{P-1}b'$, $|h|^{\frac{2P}{Q}-2}|h'|^2 = P^2|b|^{\frac{2P}{Q}-2}|b'|^2$

$$\begin{aligned} \int_{B_R \cap \mathbb{C}_+} |Du|^2 &\geq \int_{B_R \cap \{\frac{\pi}{4} \leq |\theta| < \frac{\pi}{2}\}} |Du|^2 = P \int_{B_R \cap \{\frac{\pi}{4} \leq |\theta| < \frac{\pi}{2}\}} \frac{2P}{Q} |b|^{\frac{2P}{Q}-2} |b'|^2 \\ &\geq P \int_{B_R \cap \{\frac{\pi}{4} \leq |\theta| < \frac{\pi}{2}\}} \frac{\varphi(z)}{r} \frac{\partial}{\partial r} |b|^{\frac{2P}{Q}}. \end{aligned}$$

(2.7.12) (i.e. $\frac{\partial}{\partial r} |b|^{\frac{2P}{Q}} = \frac{2P}{Q} \frac{\varphi(z)}{r} |b|^{\frac{2P}{Q}}$) and (2.7.16) show that $\frac{\partial}{\partial r} |b|^{\frac{2P}{Q}}(re^{i\theta}) \geq 0$ for $\frac{\pi}{4} \leq |\theta| \leq \frac{\pi}{2}$, $0 < r < R$, and $R > 0$ sufficient small. Hence we apply the 1-dimensional mean value theorem to deduce that to every $\frac{\pi}{4} \leq |\theta| \leq \frac{\pi}{2}$ there is $0 < r_\theta \leq R$ with

$$\begin{aligned} \int_{B_R \cap \{\frac{\pi}{4} \leq |\theta| < \frac{\pi}{2}\}} \frac{\varphi(z)}{r} \frac{\partial}{\partial r} |b|^{\frac{2P}{Q}} &= \int_{\frac{\pi}{4} \leq |\theta| \leq \frac{\pi}{2}} \varphi(r_\theta e^{i\theta}) \int_0^R \frac{\partial}{\partial r} |b|^{\frac{2P}{Q}}(re^{i\theta}) dr d\theta \\ &\geq \left(\alpha R^{-\alpha} \cos(\alpha \frac{\pi}{2}) - \frac{1}{\tanh(\frac{\pi}{4})} \right) \int_{\frac{\pi}{4} \leq |\theta| \leq \frac{\pi}{2}} |b|^{\frac{2P}{Q}}(Re^{i\theta}) d\theta. \end{aligned}$$

(Again we can avoid measurability questions using the bound (2.7.15), nonetheless compare the previous remark 2.7.5.) Recall (2.7.16) to deduce (iii) in total since for $R > 0$ sufficient small

$$I(u, 0, R) \geq \frac{P}{Q} \left(\alpha R^{-\alpha} \cos(\alpha \frac{\pi}{2}) - \frac{1}{\tanh(\frac{\pi}{4})} \right) \rightarrow \infty \quad (R \rightarrow 0).$$

□

Proof of corollary 2.7.8. We claim that for the choice $f(z) = e^{-F(z)}$ of lemma 2.4.1 with a fixed $0 < s < 1$ the minimizer $u(z) = \sum_{\substack{v \in \mathbb{C} \\ v^Q = f(z)}} \|v\|$ has the desired properties.

(i) follows as before by similar arguments presented in the proof to corollary 2.7.5 and so we omit the details.

(ii) corresponds with $f(z) \neq 0$ for all $z \in \mathbb{C}_+$.

(iii) remains to be proven. We define

$$\begin{aligned} R_n &= |E_{n,\cdot}| + \frac{2}{3} (|E_{n-1,\cdot}| - 2|E_{n,\cdot}|) = \frac{2}{3}|E_{n-1,\cdot}| - \frac{1}{3}|E_{n,\cdot}| = \frac{1}{3}(2^{1+\frac{1}{s}} - 1)2^{-\frac{n}{s}} \\ \underline{R}_n &= |E_{n,\cdot}| + \frac{1}{3} (|E_{n-1,\cdot}| - 2|E_{n,\cdot}|) = \frac{1}{3}|E_{n-1,\cdot}| + \frac{1}{3}|E_{n,\cdot}| = \frac{1}{3}(2^{\frac{1}{s}} + 1)2^{-\frac{n}{s}} \end{aligned}$$

and set $\delta = \frac{1}{3}(2^{\frac{1}{s}} - 2) > 0$. We will show that (iii) holds for the countable set $\{y_\tau\}_{\tau \in \mathcal{I}}$ and the sequence R_n .

Let y_{τ_0} be given and fixed from now on. Set

$$\mathcal{I}_0 = \{\tau \in \mathcal{I} : y_\tau = y_{\tau_0}\};$$

hence for any $! \exists k_0 \in \mathbb{N}$ s.t. $\forall \tau = (k, l)$ with $k < k_0$, $y_\tau \neq y_{\tau_0}$ and $\forall k > k_0$ $! \exists \tau = (k, l) \in \mathcal{I}_0$. We may assume that $\tau_0 = (k_0, l_0)$. We partition $\mathcal{I} \setminus \mathcal{I}_0$ as follows:

$$\mathcal{I}_1 = \{\tau \in \mathcal{I} : y_\tau \notin E_{\tau_0}\}$$

and for any $\tau = (k, l) \in \mathcal{I}_0 \setminus \{\tau_0\}$ (i.e. l is odd and $k > k_0$) set

$$\mathcal{I}_\tau = \{\tau' \in \mathcal{I} : y_{\tau'} \in E_{k, l+1} \cap E_{\tau_0}\}.$$

Observe that then for each such $\tau = (k, l) \in \mathcal{I}_0$, $\tilde{k} \geq k > k_0$ one has

$$|\{\tau' = (k', l') \in \mathcal{I}_\tau : k' = \tilde{k}\}| = 2^{\tilde{k}-k}.$$

Define

$$\varphi(z + iy_{\tau_0}) = \Re(-F'(z)(z + iy_{\tau_0})).$$

To simplify notation we will set $r = r_{\tau_0}$, $\theta = \theta_{\theta_0}$ i.e. $z + iy_{\tau_0} = re^{i\theta}$. (2.7.12) in our case corresponds to

$$(2.7.17) \quad \frac{\partial}{\partial r} |f|^{\frac{2}{Q}} = \frac{2}{Q} \frac{\varphi(re^{i\theta})}{r} |f|^{\frac{2}{Q}}.$$

Recall from lemma 2.4.1 that $\frac{-1}{\alpha} F'(z)(z + iy_{\tau_0}) = \sum_{\tau \in \mathcal{I}} a_k(z + iy_\tau)^{-\alpha-1}(z + iy_{\tau_0})$ converging absolutely and $\Re((z + iy_\tau)^{-\alpha-1}(z + iy_{\tau_0})) = r_\tau^{-\alpha-1} r \cos((\alpha+1)\theta_\tau - \theta)$. For $\tau \in \mathcal{I}_0$ we have $z + iy_\tau = z + iy_{\tau_0} = re^{i\theta}$ and so

$$\Re\left(\sum_{\tau \in \mathcal{I}_0} a_k(z + iy_\tau)^{-\alpha-1}(z + iy_{\tau_0})\right) = r^{-\alpha} \cos(\alpha\theta) \sum_{\tau \in \mathcal{I}_0} a_k \geq c_0 r^{-\alpha}$$

with $c_0 = \cos(\alpha\frac{\pi}{2}) \sum_{k=k_0}^{\infty} a_k > 0$.

For $\tau \in \mathcal{I}_1$, $0 < r < R$, $R > 0$ sufficient small we have $r_\tau \geq \delta|E_{k_0, \cdot}|$ because $r_\tau \geq |E_{k_0-1, \cdot}| - 2|E_{k_0, \cdot}| - r$. Therefore we found

$$\Re\left(\sum_{\tau \in \mathcal{I}_1} a_k(z + iy_\tau)^{-\alpha-1}(z + iy_{\tau_0})\right) \geq -(\delta|E_{k_0, \cdot}|)^{-\alpha-1} r \sum_{\tau \in \mathcal{I}_1} a_k \geq -c_1 r$$

In the rest of the argument we restrict us to $\underline{R}_n \leq r \leq R_n$ and $n > N$ for some large $N \in \mathbb{N}$. If $\tau = (k, l) \in \mathcal{I}_0$ with $k_0 < k \leq n$ and $\tau' \in \mathcal{I}_\tau$ then $r_\tau \geq |y_{\tau'} - y_\tau| - r \geq |E_{k-1, \cdot}| - |E_{k, \cdot}| - R_n \geq \delta|E_{k, \cdot}|$, so that

$$\begin{aligned} \sum_{\substack{\tau=(k,l) \in \mathcal{I}_0 \\ k_0 < k \leq n}} \sum_{\tau' \in \mathcal{I}_\tau} a_{k'} r_{\tau'}^{-\alpha-1} r \cos((\alpha+1)\theta_{\tau'} - \theta) &\geq - \sum_{k_0 < k \leq n} (\delta|E_{k, \cdot}|)^{-\alpha-1} r \sum_{k'=k}^{\infty} \frac{2^{-k}}{(k')^2} \\ &\geq -\frac{r}{\delta^{\alpha+1}} \sum_{k_0 < k \leq n} \frac{M^k}{k-1} \geq -r \frac{M^{n+1} - M^{k_0+1}}{\delta^{\alpha+1} k_0 (M-1)} \geq -\frac{c'_2}{k_0} r M^n \end{aligned}$$

where $M = (2^{\frac{\alpha+1}{s}-1}) > 1$. If $\tau = (k, l) \in \mathcal{I}_0$ with $n < k$ and $\tau' \in \mathcal{I}_\tau$ then $r_\tau \geq r - |y_{\tau'} - y_\tau| \geq \underline{R}_n - |E_{k-1, \cdot}| \geq \underline{R}_n - |E_{n, \cdot}| = \delta|E_{n, \cdot}|$ hence

$$\begin{aligned} \sum_{\substack{\tau=(k,l) \in \mathcal{I}_0 \\ n < k}} \sum_{\tau' \in \mathcal{I}_\tau} a_{k'} r_{\tau'}^{-\alpha-1} r \cos((\alpha+1)\theta_{\tau'} - \theta) &\geq -(\delta|E_{n, \cdot}|)^{-\alpha-1} r \sum_{k=n+1}^{\infty} \sum_{k'=k}^{\infty} \frac{2^{-k}}{(k')^2} \\ &\geq -(\delta|E_{n, \cdot}|)^{-\alpha-1} r \sum_{k=n+1}^{\infty} \frac{2^{-k}}{k-1} \geq -r \frac{1}{\delta^{\alpha+1} n} M^n = -\frac{c''_2}{n} r M^n. \end{aligned}$$

Summarizing for $\underline{R}_n \leq r \leq R_n$ and $n \geq N = N(k_0)$, we have

$$(2.7.18) \quad \frac{1}{\alpha} \varphi(re^{-i\theta}) \geq r^{-\alpha} \left(c_0 - c_1 r^{1+\alpha} - \left(\frac{c'_2}{k_0} + \frac{c''_2}{n} \right) M^n r^{1+\alpha} \right) \geq \frac{c_0}{2} r^{-\alpha}$$

because $M^n r^{1+\alpha} \leq M^n R_n^{1+\alpha} = \left(\frac{1}{3} (2^{1+\frac{1}{s}} - 1) 2^{-\frac{n}{s}} \right)^{1+\alpha} 2^{-n} \rightarrow 0$ (as $n \rightarrow \infty$).

(2.7.17) and (2.7.18) gives for $\underline{R}_n \leq r \leq R_n$

$$\frac{\partial}{\partial r} \ln(|f|^{\frac{2}{Q}}(-iy_{\tau_0} + r^{i\theta})) = \frac{2\alpha}{Q} \varphi(re^{i\theta}) \geq \frac{c_0\alpha}{Q} r^{-\alpha}$$

or integrated

$$(2.7.19) \quad \ln \left(\frac{|f|^{\frac{2}{Q}}(R_n e^{i\theta})}{|f|^{\frac{2}{Q}}(\underline{R}_n e^{i\theta})} \right) \geq c R_n^{-\alpha}$$

with $c = \frac{c_0}{Q} \left(\left(\frac{R_n}{\underline{R}_n} \right)^\alpha - 1 \right) > 0$ (independent of n).

Now we combine the just established with (2.7.11)

$$\begin{aligned} \int_{B_{R_n}(-iy_{\tau_0}) \cap \mathbb{C}_+} |Du|^2 &\geq \int_{\{\underline{R}_n \leq |z+iy_{\tau_0}| \leq R_n\} \cap \mathbb{C}_+} |Du|^2 \\ &= \frac{2}{Q} \int_{\{\underline{R}_n \leq r \leq R_n\} \cap \mathbb{C}_+} |-F'|^2 |f|^{\frac{2}{Q}} \geq \int_{\{\underline{R}_n \leq r \leq R_n\} \cap \mathbb{C}_+} \frac{2}{Q} \frac{\varphi(re^{i\theta})}{r} \frac{\partial}{\partial r} |f|^{\frac{2}{Q}}. \end{aligned}$$

(2.7.17) and (2.7.18) show that $\frac{\partial}{\partial r} |f|^{\frac{2}{Q}} > 0$ for $\underline{R}_n \leq r \leq R_n$. We apply as before the 1-dimensional mean value theorem to deduce that to every $|\theta| \leq \frac{\pi}{2}$ there is $0 < r_\theta \leq R$ with

$$\begin{aligned} \int_{\{\underline{R}_n \leq r \leq R_n\} \cap \mathbb{C}_+} \frac{2}{Q} \frac{\varphi(re^{i\theta})}{r} \frac{\partial}{\partial r} |f|^{\frac{2}{Q}} &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \varphi(r_\theta e^{i\theta}) \int_{\underline{R}_n}^{R_n} \frac{\partial}{\partial r} |f|^{\frac{2}{Q}} dr d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \varphi(r_\theta e^{i\theta}) \left(|f|^{\frac{2}{Q}}(-iy_{\tau_0} + R_n e^{i\theta}) - |f|^{\frac{2}{Q}}(-iy_{\tau_0} + \underline{R}_n e^{i\theta}) \right) d\theta \\ &\geq \frac{\alpha c_0}{2} R_n^{-\alpha} \left(1 - e^{-c R_n^{-\alpha}} \right) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |f|^{\frac{2}{Q}}(-iy_{\tau_0} + R_n e^{i\theta}) d\theta \end{aligned}$$

(With the same observations as before, we can avoid measurability questions by (2.7.18).) We used in the last line (2.7.18) and (2.7.19). Finally remembering (2.7.10) we conclude (iii) since we found

$$I(u, -iy_{\tau_0}, R_n) \geq \frac{\alpha c_0}{2Q} R_n^{-\alpha} \left(1 - e^{-c R_n^{-\alpha}} \right) \rightarrow \infty \quad (\text{as } n \rightarrow \infty).$$

□

2.7.3. Unique continuation. Consider an elliptic operator L in divergence form

$$(2.7.20) \quad Lu = D_i(a^{ij}(x)D_j u) + b^i(x)D_i u + c(x)u.$$

A function $u \in L^2_{loc}(\Omega)$ is said to vanish of infinite order at a point $x_0 \in \bar{\Omega}$ if

$$(2.7.21) \quad \int_{B_R(x_0) \cup \Omega} |u|^2 = O(R^k) \text{ for every } k \in \mathbb{N}.$$

An elliptic operator L as in (2.7.20) is said to have the strong unique continuation property in Ω if the only $H^{1,2}_{loc}(\Omega)$ solution of $Lu = 0$ on Ω which vanishes of infinite order at a point $x_0 \in \Omega$ is $u \equiv 0$.

N. Garofalo, F. Lin showed in [7, Theorem 1.1] that L has the unique continuation property under certain assumptions on the regularity and ellipticity of the coefficients $a^{ij}(x), b^i(x), c(x)$. They are able to deduce their result proving a doubling

theorem like the following, which we prove using the frequency function. (The quoted version can be found in [4, Theorem 6.1])

Theorem 2.7.9. *Let L as in (2.7.20) with $a^{ij}(x)$ symmetric, uniformly elliptic and Lipschitz, $b^i(x), c(x)$ continuous, then if $u \in H_{loc}^{1,2}(B_{2R_0}(x_0))$ nonconstant solves $Lu = 0$ on $B_{2R_0}(x_0)$ then there exists $0 < R = R(a^{ij}, b^i, c, x_0) < R_0$ and $\bar{d} = \bar{d}(a^{ij}, b^i, c, x_0, u) > 0$ s.t.*

$$\int_{B_{2r}(x_0)} u^2 \leq 2^{2\bar{d}} \int_{B_r(x_0)} u^2 \quad \forall 0 < r < R$$

A consequence of lemma 2.4.1 is that a strong unique continuation theorem fails for boundary points.

Example 2.7.6. Given $0 < s \leq 1$ there exists $u \in C^\infty(\overline{\mathbb{R}_+^2})$, $u \neq 0$ with

$$\Delta u = 0 \text{ on } \mathbb{R}_+^2 \text{ (i.e. harmonic)}$$

and a set $E_s \subset \partial\mathbb{R}_+^2$ with $\mathcal{H}^s(E_s) = 1$ ($0 < s < 1$), $\dim_{\mathcal{H}}(E_s) = 1$ ($s = 1$) such that u vanishes to infinite order for all $z \in -iE_s$.

Observe that Δ satisfies the conditions of theorem 2.7.9 and therefore has the strong unique continuation property in the interior of \mathbb{R}_+^2 .

Proof of example 2.7.6. Let $0 < s \leq 1$ be given and f the related holomorphic function of lemma 2.4.1. Since f is C^∞ on $\overline{\mathbb{C}_+}$ (2.6.4) and $\overline{\mathbb{C}_+}$ convex we have by 1-dimensional analysis

$$(2.7.22) \quad f(z) = \sum_{l=1}^{k-1} \frac{1}{l!} f^{(l)}(z_0)(z-z_0)^l + \frac{1}{(k-1)!} \int_0^1 (1-s)^{k-1} f^{(k)}(z_0 + s(z-z_0))(z-z_0)^k ds.$$

The function

$$u(z) = \Re(f(z))$$

is harmonic and non-constant on \mathbb{R}_+^2 , C^∞ on $\overline{\mathbb{R}^2}$ and has the desired property since for $z_0 \in -iE_s$, $f^{(l)}(z_0) = 0 \forall l$ and therefore by (2.7.22)

$$|u(z)| \leq \frac{1}{k!} \sup_{w \in \overline{\mathbb{C}_+} \cap B_1(z_0)} |f^{(k)}(w)| |z-z_0|^k \text{ for all } z \in \overline{\mathbb{C}_+} \cap B_1(z_0).$$

This implies that u satisfies (2.7.21). □

Part 3. Partial Hölder continuity for Q -valued energy minimizing maps

3.8. INTRODUCTION

This part addresses an interior partial regularity result for Q -valued maps between $\Omega \subset \mathbb{R}^N$ open ($N \geq 2$) and a smooth, compact Riemannian manifold \mathcal{N} .

Multivalued maps with focus on Dirichlet integral minimizing maps have been introduced by F. Almgren in his fundamental work [1]. C. De Lellis and E. Spadaro gave a modern revision of it in [12]. They considered maps valued in some \mathbb{R}^m . Hölder continuity in the interior for minimizers is an outcome of Almgren's original work. Furthermore a minimizer $u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^m))$ is the "superposition" of "classical", single valued harmonic functions outside a singular set $\text{sing}(u)$, with Hausdorff dimension not exceeding $N - 2$ in the following sense: $y \notin \text{sing}(u)$, $U_y \subset \Omega$ open neighborhood of y , $u_i : U_y \rightarrow \mathbb{R}^m$ ($i = 1, \dots, Q$) harmonic with $u(x) = \sum_{i=1}^Q \llbracket u_i(x) \rrbracket \forall x \in U_y$.

The aim of this note is to extend the theory of harmonic maps from the single valued to the multivalued equivalent i.e. Q -valued maps into a smooth, compact Riemannian manifold locally minimizing the Dirichlet integral. The interior Hölder regularity for single valued minimizing harmonic maps has been known since the work of R. Schoen and K. Uhlenbeck, [19]. In this note we give an interior partial Hölder-regular result for multivalued maps minimizing locally the Dirichlet energy. Our strategy is inspired by the methods of S. Luckhaus, see [16]. We are able to show:

Theorem 3.8.1. *There is a constant $\alpha = \alpha(N, Q) > 0$ with the property that, if $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is open, $\mathcal{N} \subset \mathbb{R}^m$ is a smooth compact n -dimensional Riemannian sub-manifold and $u \in W_{loc}^{1,2}(\Omega, \mathcal{A}_Q(\mathcal{N}))$ is locally minimizing the Dirichlet energy, then there exists $\Omega' \subset \Omega$ open, such that $u \in C^{0,\alpha}(\Omega')$ and $\Omega \setminus \Omega'$ has at most Hausdorff dimension $N - 3$.*

For single valued harmonic maps one has the following sharper result: if Ω, \mathcal{N} as above and $v \in W_{loc}^{1,2}(\Omega, \mathcal{N})$ is locally Dirichlet minimizing, then $\exists \Omega' \subset \Omega$ with $\dim_{\mathcal{H}}(\Omega \setminus \Omega') \leq N - 3$ and $v \in C^\infty(\Omega')$. The main difference is that the C^∞ regularity for single valued maps is replaced by Hölder-regularity in the multivalued setting. Furthermore we want to mention that in the single valued case the result above can be sharpened when the target manifold satisfies some special structural assumptions.

A pressing open question in the Q -valued case is to give a more detailed description of the singular set in the interior of the Hölder regular set Ω' of theorem 3.8.1: How small is the set $\text{sing}(u) \cap \Omega'$ s.t. u can be written as a "superposition" of "classical", single valued harmonic maps. One should compare it to the corresponding result of a minimizers u mapping into $\mathcal{A}_Q(\mathbb{R}^m)$ mentioned above. Another possible extension is to consider maps minimizing the p -Dirichlet integral in the spirit of S. Luckhaus [16].

This part is organized as follows: after fixing some notation and definitions in section 3.9, we extend the "classical" variational equations and monotonicity formula to the multivalued setting in section 3.10. Section 3.11 collects some tools to derive a compactness result for minimizers in section 3.12 and the interior partial Hölder-continuity result in section 3.13. Section 3.14 uses the obtained to conclude the estimate on the size of the Hölder-singular-set following classical lines. The

appendix, A, contains an intrinsic proof to the "classical" Luckhaus' lemma concerning the extension of a map Sobolev map defined on the boundary of an annulus $\partial(B_1 \setminus B_{1-\lambda})$ into the interior.

3.9. DEFINITION OF ENERGY MINIMIZING MAPS

Suppose $\Omega \subset \mathbb{R}^N$ open, $N \geq 2$ and \mathcal{N} is a smooth compact n -dimensional Riemannian manifold isometrically embedded in some \mathbb{R}^m .

$\mathcal{A}_Q(\mathcal{N})$ denotes corresponding to $\mathcal{A}_Q(\mathbb{R}^m)$ classical metric space of unordered Q -tuples taking values in \mathcal{N} instead of the whole \mathbb{R}^m .

Definition 3.9.1. (i) $W_{loc}^{1,2}(\Omega, \mathcal{A}_Q(\mathcal{N}))$ is the set of $u \in W_{loc}^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^m))$ s.t. $u(x) \in \mathcal{A}_Q(\mathcal{N})$ for a.e. $x \in \Omega$. Since \mathcal{N} is assumed to be compact we have $W_{loc}^{1,2}(\Omega, \mathcal{A}_Q(\mathcal{N})) \subset L^\infty(\Omega, \mathcal{A}_Q(\mathbb{R}^m))$.

(ii) For any $\overline{B_R(y)} \subset \Omega$ we define the energy $E(u, B_R(y))$ as

$$(3.9.1) \quad E(u, B_R(y)) = R^{2-N} \int_{B_R(y)} |Du|^2.$$

(iii) We call $u \in W_{loc}^{1,2}(\Omega, \mathcal{A}_Q(\mathcal{N}))$ a *local minimizer*, or just a *minimizer*, if for any $\overline{B_R(y)} \subset \Omega$ and $v \in W_{loc}^{1,2}(\Omega, \mathcal{A}_Q(\mathcal{N}))$ satisfying $u = v$ in a neighbourhood of $\partial B_R(y)$ we have

$$E(u, B_R(y)) \leq E(v, B_R(y)).$$

We want to study the regularity of such energy minimizing maps. For this purpose we define the regular and singular set.

Definition 3.9.2. A Q -valued map $u \in W_{loc}^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^m))$ is called regular at a point $y \in \Omega$ if there exists a neighborhood U of y and Q smooth maps $u_i : U \rightarrow \mathbb{R}^m$ s.t.

$$u(x) = \sum_{i=1}^Q \llbracket u_i(x) \rrbracket \text{ for a.e. } x \in U$$

and either $u_i(x) \neq u_j(x)$ for all $x \in U$ or $u_i \equiv u_j$.

We define the open set

$$(3.9.2) \quad \text{reg } u = \{y \in \Omega : y \text{ is a regular point of } u\}.$$

The singular set of u is the relative closed set $\text{sing } u = \Omega \setminus \text{reg } u$.

Remark 3.9.3. If $y \in \text{reg } u$ and $u(x) = \sum_{i=1}^Q \llbracket u_i(x) \rrbracket$ on a neighborhood U each $u_i : U \rightarrow \mathcal{N}$ has to be a smooth energy minimizing i.e. a harmonic map in the classical sense.

For our purpose it is helpful to define a certain subset of the singular set.

Definition 3.9.4. If $u \in W_{loc}^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^m))$ then the Hölder regular set of u is

$$(3.9.3) \quad \text{reg}_H u = \{y \in \Omega : u \text{ is Hölder continuous in a neighborhood of } y\},$$

and the Hölder singular set is $\text{sing}_H u = \Omega \setminus \text{reg}_H u$.

Just by definition we have

$$\text{reg } u \subset \text{reg}_H u \text{ and } \text{sing}_H u \subset \text{sing } u.$$

Remark 3.9.5. If $\mathcal{N} = \mathbb{R}^n$ then $\text{sing}_H u = \emptyset$ or $\text{reg}_H u = \Omega$ for any minimizing $u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ as a consequence of the internal Hölder regularity result on \mathbb{R}^n , e.g. [12, Theorem 0.9]. $\text{sing}_H u$ is not empty in general, since $\text{sing}_H u$ is already known to be non-empty in certain cases of classical single-valued energy minimizing maps.

3.10. THE VARIATIONAL EQUATIONS AND MONOTONICITY FORMULAS

Suppose $u \in W_{loc}^{1,2}(\Omega, \mathcal{A}_Q(\mathcal{N}))$ is a energy minimizing map and $\overline{B_R(y)} \subset \Omega$. Suppose $\{u_t\}_{t \in]-\delta, \delta[}$ is a C^1 family of maps in $W^{1,2}(B_R(y), \mathcal{A}_Q(\mathcal{N}))$ s.t. $u_s = u$ in a neighborhood of $\partial B_R(y)$ for all t and $u_0 = u$ then due to minimality of u we must have

$$(3.10.1) \quad \left. \frac{d}{dt} \right|_{t=0} E(u_t, B_R(y)) = 0.$$

There two natural classes of variations, inner and outer once.

3.10.0.1. *inner variations.* Let $\Phi_t(x) = x + tX(x) + o(t)$ be the flow generated by an arbitrary vector field $X = (X^1, \dots, X^N) \in C_c^1(B_R(y), \mathbb{R}^N)$. Since $\Phi_t(x) = x$ close to $\partial B_R(y)$ $v_t(x) = u \circ \Phi_t^{-1}(x)$ defines a C^1 -family of competitors to u . Standard calculations give

$$\begin{aligned} D\Phi_t^{-1} \circ \Phi_t &= (D\Phi_t)^{-1} = \mathbf{1} - tDX + o(t) \\ \det(D\Phi_t) &= 1 + t \operatorname{div}(X) + o(t) \end{aligned}$$

($\mathbf{1}$ denotes the identity map on \mathbb{R}^N) so that

$$\begin{aligned} |Dv_t|^2 \circ \Phi_t &= \sum_{l=1}^Q |Du_l D\Phi_t^{-1} \circ \Phi_t|^2 = \sum_{l=1}^Q |Du_l (\mathbf{1} - tDX + o(t))|^2 \\ &= \sum_{l=1}^Q |Du_l|^2 - 2t \sum_{l=1}^Q \langle Du_l : Du_l DX \rangle + o(t). \end{aligned}$$

Integrating we get

$$\begin{aligned} \int_{B_R(y)} |Dv_t|^2 &= \int_{B_R(y)} |Dv_t|^2 \circ \Phi_t |\det D\Phi_t| \\ &= \int_{B_R(y)} |Du|^2 + t \int_{B_R(y)} |Du|^2 \operatorname{div}(X) - 2 \sum_{l=1}^Q \langle Du_l : Du_l DX \rangle + o(t). \end{aligned}$$

Because of (3.10.1) we necessarily have

$$(3.10.2) \quad 0 = \int_{B_R(y)} \left(|Du|^2 \delta_{ij} - 2 \sum_{l=1}^Q \langle D_i u_l : D_j u_l \rangle \right) D_i X^j.$$

Before we consider the second class, the outer variations, it is useful to set up some terminology and recall some facts about the nearest point projection.

$\mathcal{N}_d = \{x : \operatorname{dist}(x, \mathcal{N}) < d\}$ defines a tubular neighbourhood around \mathcal{N} for any $d > 0$. Given $p \in \mathcal{N}$ and a vector $X \in \mathbb{R}^m$, X^\top denotes the orthogonal projection of X onto $T_p \mathcal{N}$; hence $X^\perp = X - X^\top$ is the orthogonal projection onto the normal space $(T_p \mathcal{N})^\perp$ at p . $A_p(X_1^\top, X_2^\top) = (D_{X_1^\top} X_2^\top)^\perp$ is the second fundamental form of \mathcal{N} at p and any vector fields $X_1, X_2 \in C(B_\epsilon(p), \mathbb{R}^m)$.

Remark 3.10.1. Since \mathcal{N} is assumed to be a smooth compact manifold it has a nearest point projection Π . Π is defined on some tubular neighbourhood \mathcal{N}_d , ($d > 0$). Beside being a smooth map i.e. $\Pi \in C^\infty(\mathcal{N}_d; \mathcal{N})$ it has the following properties:

- (i) $|x - \Pi(x)| = \operatorname{dist}(x, \mathcal{N}) < |x - p|$ for all $x \in \mathcal{N}_d$ and $p \in \mathcal{N} \setminus \{\Pi(x)\}$;
- (ii) $D\Pi(p)X = X^\top$ for $p \in \mathcal{N}$ and any vector $X \in \mathbb{R}^m$;

(iii) for $p \in \mathcal{N}$ and any vectors $X_i \in \mathbb{R}^N (i = 1, 2, 3)$

$$(3.10.3) \quad A_p(X_1^\top, X_2^\top) = D^2\Pi(p)(X_1^\top, X_2^\top)$$

$$(3.10.4) \quad X_1 D^2\Pi(p)(X_2, X_3) = \sum_{\sigma \in \mathcal{P}_3} X_{\sigma(1)}^\perp D^2\Pi(p)(X_{\sigma(2)}^\top, X_{\sigma(3)}^\top)$$

(iv) for any $x \in \mathcal{N}_d$ and any vector $X \in \mathbb{R}^m$ we have

$$(1 - 2 \operatorname{dist}(x, \mathcal{N}) \|A_{\Pi(x)}\|) |D\Pi(x)X|^2 \leq |X|^2.$$

Although all of these are classical, we give their proofs expect for showing existence and smoothness of Π , that can be found for example in [21, 2.12.3 Theorem 1].

(i) is the defining property of Π as nearest point projection.

(ii) For $X \in \mathbb{R}^m$ given, we may write $X = X^\top + X^\perp$. Take a curve $\gamma :]-\delta, \delta[\rightarrow N$ with $\gamma(0) = p$ and $\gamma'(0) = X^\top$. Since $\Pi(\gamma(t)) = \gamma(t)$ we have

$$|\Pi(p + tX^\top) - \gamma(t)| \leq |p + tX^\top - \Pi(p + tX^\top)| + |p + tX^\top - \gamma(t)| \leq 2|p + tX^\top - \gamma(t)|$$

that is of order $o(t)$ and so $D\Pi_p X^\top = X^\top$. Since $\Pi(p + tX^\perp) = p$ for all t we conclude

$$D\Pi(p)(X) = \frac{d}{dt} \Big|_0 \Pi(p + tX^\top + tX^\perp) = \frac{d}{dt} \Big|_0 \Pi(p + tX^\top) + \frac{d}{dt} \Big|_0 \Pi(p + tX^\perp) = X^\top;$$

(iii) Let $X_2 \in C^\infty(\mathcal{N}_d, \mathbb{R}^m)$ and let γ be a curve in \mathcal{N} with $\gamma(0) = p, \gamma'(0) = X_1^\top$. Differentiating $X_2^\top \circ \gamma(t) = D\Pi(\gamma(t))X_2(\gamma(t))$ we deduce

$$D_{X_1^\top}(X_2^\top) = (D_{X_1^\top} X_2)^\top + D^2\Pi(p)(X_1^\top, X_2)$$

with the particular choices $X_2 = X_2^\top, X_2 = X_2^\perp$ we reach

$$D^2\Pi(p)(X_1^\top, X_2^\top) = D_{X_1^\top}(X_2^\top) - (D_{X_1^\top} X_2^\top)^\top = (D_{X_1^\top} X_2^\top)^\perp = A_p(X_1^\top, X_2^\top),$$

$$D^2\Pi(p)(X_1^\top, X_2^\perp) = -(D_{X_1^\top} X_2^\perp)^\top.$$

Recall that $\langle X_2^\top, X_3^\perp \rangle = 0$ implies $0 = \langle D_{X_1^\top} X_2^\top, X_3^\perp \rangle + \langle X_2^\top, D_{X_1^\top} X_3^\perp \rangle$. Additionally one has $D^2\Pi(p)(X_2^\perp, X_3^\perp) = 0$ since $p = \Pi(p) = \Pi(p + sX_2^\perp + tX_3^\perp)$ for all s, t . A short calculation give the desired conclusion (3.10.4).

(iv) Let $\gamma :]-\delta, \delta[\rightarrow \mathcal{N}_d$ be any C^2 curve in the tubular neighborhood of \mathcal{N} with $\gamma(0) = x$ and $\gamma'(0) = X$ e.g. $\gamma(t) = x + tX$. Define $\tilde{\gamma}(t) = \Pi(\gamma(t))$ the corresponding C^2 curve on \mathcal{N} . Hence $\tilde{\gamma}'(0) = D\Pi(x)X$. Set $\nu(t) = \gamma(t) - \tilde{\gamma}(t)$ i.e. $|\nu(t)| = \operatorname{dist}(\gamma(t), \mathcal{N})$ and hence $0 = \langle \tilde{\gamma}'(t), \nu(t) \rangle$. Differentiating we obtain

$$\langle \tilde{\gamma}'(t), \nu'(t) \rangle = -\langle \tilde{\gamma}''(t), \nu(t) \rangle = -\langle A_{\tilde{\gamma}(t)}(\tilde{\gamma}'(t), \tilde{\gamma}'(t)), \nu(t) \rangle.$$

Furthermore we find

$$\begin{aligned} |X|^2 &= |\tilde{\gamma}'(0)|^2 + 2\langle \tilde{\gamma}'(0), \nu'(0) \rangle + |\nu'(0)|^2 \\ &\geq |\tilde{\gamma}'(0)|^2 - 2\langle A_{\tilde{\gamma}(0)}(\tilde{\gamma}'(0), \tilde{\gamma}'(0)), \nu(0) \rangle \\ &\geq |\tilde{\gamma}'(0)|^2 - 2 \operatorname{dist}(x, \mathcal{N}) \|A_{\Pi(x)}\| |\tilde{\gamma}'(0)|^2. \end{aligned}$$

Now we are ready to consider outer variations.

3.10.0.2. *Outer variations.* Let $Y = (Y^1, \dots, Y^n) \in C^1(B_R(y) \times \mathbb{R}^n, \mathbb{R}^n)$ be an arbitrary vector field with $Y(x, z) = 0$ for x close to $\partial B_R(y)$. Set $\Psi_t(x, z) = z + tY(x, z)$. For sufficiently small t we obtain a C^1 family of competitors setting

$$v_t(x) = \Pi(\Psi_t(x, u(x))).$$

$\Pi(p + tY(x, p)) = p + t \int_0^1 D_i \Pi(p + stY(x, p)) Y^i(x, p) ds \quad \forall p \in \mathcal{N}$ and small t and apply the chain rule ([12, Proposition 1.12]) for the 1-jet of v_t to conclude

$$\begin{aligned} J\mathcal{V}_{t,x}(y) &= \sum_{l=1}^Q [u_l(x) + o(t) + \\ &\quad t (D\Pi(u_l(x)) D_i Y(x, u_l(x)) + D^2 \Pi(u_l(x)) (D_i u_l(x), Y(x, u_l(x)))) (y^i - x^i)]. \\ \langle D_i u_l, D^2 \Pi(u_l) (D_i u_l, Y) \rangle &= \langle Y, A_{u_l} (D_i u_l, D_i u_l) \rangle \text{ as seen in remark 3.10.1 (iii), since} \\ D_i u_l(x) \in T_{u_l(x)} \mathcal{N} \text{ for a.e. } x. \end{aligned} \quad (3.10.1)$$

$$(3.10.5) \quad 0 = \int_{B_R(y)} \left(\sum_{i=1}^N \sum_{l=1}^Q \langle D_i u_l, D_i Y(x, u_l) \rangle + \langle A_{u_l} (D_i u_l, D_i u_l), Y(x, u_l) \rangle \right).$$

3.10.0.3. *Monotonicity formulas:* Let $u \in W_{loc}^{1,2}(\Omega, \mathcal{A}_Q(\mathcal{N}))$ be an energy minimizing map and $\overline{B_R(y)} \subset \Omega$. For a.e. $0 < r \leq R$ we have

$$(3.10.6) \quad \int_{B_r(y)} \left(\sum_{i=1}^N \sum_{l=1}^Q \langle D_i u_l, D_i u_l \rangle + \langle A_{u_l} (D_i u_l, D_i u_l), u_l \rangle \right) = \int_{\partial B_r(y)} \sum_{l=1}^Q \langle u_l, \frac{\partial u_l}{\partial r} \rangle,$$

and

$$(3.10.7) \quad (2 - N) \int_{B_r(y)} |Du|^2 = 2r \int_{\partial B_r(y)} \left| \frac{\partial u}{\partial r} \right|^2 - r \int_{\partial B_r(y)} |Du|^2.$$

To conclude these two identities recall the following general fact from analysis: if $a = (a^1, \dots, a^N) \in L^1(B_R(y), \mathbb{R}^N)$, $f \in L^1(B_R(y), \mathbb{R})$ satisfies

$$\int_{B_R(y)} a^i D_i \varphi = \int_{B_R(y)} f \varphi \quad \forall \varphi \in C_c^\infty(B_R(y))$$

then

$$(3.10.8) \quad \int_{B_r(y)} a^i D_i \phi - f \phi = \int_{\partial B_r(y)} \phi \langle a, \nu \rangle \quad \forall \phi \in C^1(\overline{B_R(y)}), \text{ a.e. } 0 < r < R.$$

(This may be checked approximating the function $\mathbf{1}_{B_r(y)}$ with smooth functions.)

To deduce (3.10.6) choose the vector field $Y(x, z) = \varphi(x)z$, $\varphi \in C_c^\infty(B_R(y), \mathbb{R})$ in the outer variation, hence $0 = \int_{B_R(y)} a^i D_i \varphi - f \varphi$ with $a^i = \sum_{l=1}^Q \langle D_i u_l, u_l \rangle$ and $-f = \left(\sum_{l=1}^Q |Du_l|^2 + \sum_{j=1}^N \langle A_{u_l} (D_j u_l, D_j u_l), u_l \rangle \right)$. Hence (3.10.6) follows from (3.10.8) with $\phi = 1$.

(3.10.7) can be checked similarly. Apply (3.10.8) for every j separately with the choice $\phi^j(x) = (x^j - y^j)$ ($D_i \phi^j = \delta_{ij}$). Take then sum in j and conclude (3.10.7).

(3.10.7) can be considered as a differential identity. If one fix some $0 < s < R$, then (3.10.7) implies that for a.e. $s \leq r \leq R$

$$\begin{aligned} \frac{d}{dr} r^{2-N} \int_{B_r(y)} |Du|^2 &= r^{1-N} (2 - N) \int_{B_r(y)} |Du|^2 + r^{2-N} \int_{\partial B_r(y)} |Du|^2 \\ &= 2r^{2-N} \int_{\partial B_r(y)} \left| \frac{\partial u}{\partial r} \right|^2 = 2 \frac{d}{dr} \int_{B_r(y) \setminus B_s(y)} |x - y|^{2-N} \left| \frac{\partial u}{\partial r} \right|^2. \end{aligned}$$

$r \mapsto \int_{B_r(y)} f$ is an absolutely continuous function for any $f \in L^1$. So we can integrate the differential identity above and conclude the classical *monotonicity formula* for $0 < s \leq r \leq R$:

$$(3.10.9) \quad r^{2-N} \int_{B_r(y)} |Du|^2 - s^{2-N} \int_{B_s(y)} |Du|^2 = 2 \int_{B_r(y) \setminus B_s(y)} |x-y|^{2-N} \left| \frac{\partial u}{\partial r} \right|^2.$$

Notice that, due to the positivity of the right side in (3.10.9) $r \mapsto E(u, B_r(y))$, is non decreasing and its limit exists.

Definition 3.10.2. We define the density function Θ_u of u on Ω by

$$(3.10.10) \quad \Theta_u(y) = \lim_{r \rightarrow 0} E(u, B_r(y)).$$

Just note that (3.10.9) reduces in the limit $s \rightarrow 0$ to

$$(3.10.11) \quad E(u, B_r(y)) - \Theta_u(y) = 2 \int_{B_r(y)} |x-y|^{2-N} \left| \frac{\partial u}{\partial r} \right|^2.$$

3.11. THE LUCKHAUS LEMMA EXTENDED TO Q -VALUED FUNCTIONS

In this section we recall a result of S. Luckhaus, [16, Lemma 1] and extend it to Q -valued functions. As for single valued maps it is an essential tool in the proof of theorem 3.8.1. We state it in a formulation due to R. Moser in [17].

Lemma 3.11.1. *There is a constant $C = C(N, m, Q)$ such that: given $0 < \lambda < \frac{1}{2}$ and $u, v \in W^{1,2}(\mathcal{S}^{N-1}, \mathcal{A}_Q(\mathbb{R}^m))$ with*

$$\int_{\mathcal{S}^{N-1}} |D_\tau u|^2 + |D_\tau v|^2 + \frac{\mathcal{G}(u, v)^2}{\epsilon^2} = K^2$$

for some $0 < \epsilon < \lambda$, then there exists $\varphi \in W^{1,2}(B_1 \setminus B_{1-\lambda}, \mathcal{A}_Q \mathbb{R}^m)$ with the following properties

(3.11.1)

$$\varphi(x) = \begin{cases} u(x), & \text{if } |x| = 1 \\ v(\frac{x}{1-\lambda}), & \text{if } |x| = 1 - \lambda \end{cases}$$

(3.11.2)

$$\int_{B_1 \setminus B_{1-\lambda}} |D\varphi|^2 \leq C\lambda \left(\int_{\mathcal{S}^{N-1}} |D_\tau u|^2 + |D_\tau v|^2 + \frac{\mathcal{G}(u, v)^2}{\lambda^2} \right) \leq C\lambda K^2$$

(3.11.3)

$\varphi(x) \in \{y \in \mathbb{R}^m : \text{dist}(y, u(\mathcal{S}^{N-1}) \cup v(\mathcal{S}^{N-1})) < a\}$ for some $a > 0$ with

$$\begin{aligned} a^2 &\leq \frac{C}{\lambda^{N-2}} \left(\int_{\mathcal{S}^{N-1}} |D_\tau u|^2 + |D_\tau v|^2 \right)^{\frac{1}{2}} \left(\int_{\mathcal{S}^{N-1}} \mathcal{G}(u, v)^2 \right)^{\frac{1}{2}} + \frac{C}{\lambda^{N-1}} \int_{\mathcal{S}^{N-1}} \mathcal{G}(u, v)^2 \\ &\leq C_\infty Q^2 \lambda^{2-N} \epsilon K^2. \end{aligned}$$

Proof. The lemma can be concluded directly from Moser's argument, see [17, Lemma 4.4] using Almgren's bilipschitz embedding ξ .

Before we deduce it from Moser's result for Q -valued function, we shortly describe how to get the estimates with K from the Moser's result. The first is just

$$\int_{\mathcal{S}^{N-1}} |D_\tau u|^2 + |D_\tau v|^2 + \frac{\mathcal{G}(u, v)^2}{\lambda^2} \leq \int_{\mathcal{S}^{N-1}} |D_\tau u|^2 + |D_\tau v|^2 + \frac{\mathcal{G}(u, v)^2}{\epsilon^2}.$$

The second follows by Cauchy's inequality:

$$\begin{aligned} & 2\epsilon \left(\int_{\mathcal{S}^{N-1}} |D_\tau u|^2 + |D_\tau v|^2 \right)^{\frac{1}{2}} \left(\int_{\mathcal{S}^{N-1}} \frac{\mathcal{G}(u, v)^2}{\epsilon^2} \right)^{\frac{1}{2}} \\ & \leq \epsilon \left(\int_{\mathcal{S}^{N-1}} |D_\tau u|^2 + |D_\tau v|^2 + \int_{\mathcal{S}^{N-1}} \mathcal{G}(u, v)^2 \right). \end{aligned}$$

To derive the result for Q -valued functions one can argue as follows: Given u, v as stated, $\xi \circ u, \xi \circ v \in W^{1,2}(\mathcal{S}^{N-1}, \mathbb{R}^{\tilde{m}})$ are admissible, and all integral quantities are comparable up to a constant $C(N, m, Q) > 0$. By [17, Lemma 4.4] there exists a single valued function $\tilde{\varphi} \in W^{1,2}(B_1 \setminus B_{1-\lambda}, \mathbb{R}^{\tilde{m}})$ that has the stated properties, replacing u by $\xi \circ u$ and v by $\xi \circ v$. Set $\varphi = \rho \circ \tilde{\varphi} \in W^{1,2}(B_1 \setminus B_{1-\lambda}, \mathcal{A}_Q(\mathbb{R}^m))$ using Almgren's retraction ρ . φ then has the desired properties, since again all integral quantities are comparable up to a constant $C(N, m, Q) > 0$. \square

For the sake of completeness we presented here the "simple" argument based on Almgren's bilipschitz embedding ξ . The appendix A discusses this result in more detail and contains an intrinsic proof.

3.12. COMPACTNESS OF ENERGY MINIMIZING MAPS

We can follow the classical argument to get as a consequence of lemma 3.11.1 a compactness result for energy minimizing maps (compare [21, section 2.9 Lemma 1]).

Lemma 3.12.1. *If $\{u(k)\} \subset W^{1,2}(\Omega, \mathcal{A}_Q(\mathcal{N}))$ is a sequence of energy minimizing with $\limsup_{k \rightarrow \infty} E(u(k), B_\rho(y)) < \infty$ for each ball $\overline{B_R(y)} \subset \Omega$, then there is a subsequence $\{u(k')\}$ and a energy minimizer $u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathcal{N}))$ s.t.*

- (i) $\mathcal{G}(u(k'), u) \rightarrow 0$ in $L^2(\Omega)$
- (ii) $\lim_{k' \rightarrow \infty} E(u(k'), B_R(y)) = E(u, B_R(y))$ for every ball $\overline{B_R(y)} \subset \Omega$.

Proof. $\|u(k)\|_{L^\infty(\Omega)} < C$ for all k because \mathcal{N} is compact by assumption. So that

$$\limsup_{k \rightarrow \infty} \int_{\Omega'} |u(k)|^2 + |Du(k)|^2 < \infty$$

for every $\Omega' \subset \subset \Omega$. By Rellich's compactness theorem for bounded sequences in $W^{1,2}$ there is a subsequence not relabeled $u(k)$ and a $u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^m))$ s.t. $\mathcal{G}(u(k), u)(x) \rightarrow 0$ in L^2 and a.e. $x \in \Omega$. Hence $u(x) \in \mathcal{A}_Q(\mathcal{N})$ for a.e. $x \in \Omega$. It remains to prove that

$$\lim_{k \rightarrow \infty} E(u(k), B_R(y)) = E(u, B_R(y)) \quad \forall \overline{B_R(y)} \subset \Omega.$$

Let be $B_R(y)$ be given, not changing notation we write $u(k)(x)$ for $u(k)(y + Rx)$, so we can assume that $B_R(y)$ is the unite ball B_1 . So $\mathcal{G}(u(k), u) \rightarrow 0$ in $L^1(B_1)$ and there is $K > 0$ with $\liminf_{k \rightarrow \infty} \int_{B_1} |Du(k)|^2 \leq K^2$.

Let $\frac{1}{2} < r_1 < 1$ and $0 < \delta < 1 - r_1$ arbitray small but fixed. Then fix $0 < \epsilon < \lambda < \frac{\delta}{3}$ s.t. if C is the constant of 3.11.1, d the size of the tubular neighborhood, then

$$C \frac{2^{N+2}}{1 - r_1} K^2 \lambda < \delta \text{ and } C \frac{2^{N+3}}{\lambda^{N-2}(1 - r_1)} \epsilon < d^2$$

For $u(k)_r(x) = u(k)(rx)$, $u_r(x) = u(rx)$, Fatou's lemma states

$$\begin{aligned} & \int_{r_1}^1 \liminf_{k \rightarrow \infty} \int_{S^{N-1}} |Du(k)_r|^2 + |Du_r|^2 + \frac{\mathcal{G}(u(k)_r, u_r)^2}{\epsilon^2} \\ & \leq \liminf_{k \rightarrow \infty} \int_{r_1}^1 r^{-N} \int_{\partial B_r} r^2 (|Du(k)|^2 + |Du|^2) + \frac{\mathcal{G}(u(k), u)^2}{\epsilon^2} \\ & \leq 2^N \liminf_{k \rightarrow \infty} \int_{B_1 \setminus B_{1-\lambda}} |Du(k)|^2 + |Du|^2 + \frac{\mathcal{G}(u(k), u)^2}{\epsilon^2} \leq 2^{N+1} K^2 \end{aligned}$$

Hence there is a radius ρ , $0 < r_1 < \rho < 1$ and a subsequence $u(k)$ not relabelled with

$$\int_{S^{N-1}} |Du(k)_\rho|^2 + |Du_\rho|^2 + \frac{\mathcal{G}(u(k)_\rho, u_\rho)^2}{\epsilon^2} < \frac{2^{N+2}}{1-r_1} K^2.$$

We apply the Luckhaus' lemma 3.11.1 to each tuple $u(k)_\rho, u_\rho$ and obtain $\tilde{\varphi}(k) \in W^{1,2}(B_1 \setminus B_{1-\lambda}, \mathcal{A}_Q(\mathbb{R}^m))$ with $\tilde{\varphi}(k)(x) = u(k)_\rho(x)$ for $|x| = 1$, $\tilde{\varphi}(k)(x) = u_\rho(\frac{x}{1-\lambda})$ for $|x| = 1-\lambda$, $\int_{B_1 \setminus B_{1-\lambda}} |D\tilde{\varphi}(k)|^2 \leq C \frac{2^{N+2}}{1-r_1} K^2 \lambda < \delta$ and

$$\tilde{\varphi}(k)(x) \in \{z : \text{dist}(z, u(k)(\partial B_\rho) \cup u(\partial B_\rho)) < a\}$$

with $a^2 \leq C \frac{2^{N+3}}{\lambda^{N-2}(1-r_1)} \epsilon < d^2$.

Therefore $\varphi(k)(x) = \Pi(\tilde{\varphi}(k)(\frac{x}{\rho}))$ is well defined and satisfies $\varphi(k)(x) = u(k)(x)$ for $|x| = \rho$, $\varphi(k)(x) = u(\frac{x}{1-\lambda})$ for $|x| = (1-\lambda)\rho$ and

$$\int_{B_\rho \setminus B_{(1-\lambda)\rho}} |D\varphi(k)|^2 \leq C \rho^{N-2} \int_{B_\rho \setminus B_{(1-\lambda)\rho}} |D\tilde{\varphi}(k)|^2 \left(\frac{x}{\rho}\right) \frac{dx}{\rho^N} \leq C \delta \rho^{N-2}.$$

Given a competitor $v \in W^{1,2}(B_\rho, \mathcal{A}_Q(\mathcal{N}))$ to u , the map

$$v(k) = \begin{cases} u(k), & \text{for } \rho \leq |x| \leq 1 \\ \varphi(k), & \text{for } (1-\lambda)\rho \leq |x| \leq \rho \\ v(\frac{x}{1-\lambda}), & \text{for } |x| \leq (1-\lambda)\rho \end{cases}$$

defines a competitor to $u(k)$. Hence by minimality of each $u(k)$ we got

$$\int_{B_1} |Du(k)|^2 \leq \int_{B_1} |Dv(k)|^2 \leq \int_{B_1 \setminus B_\rho} + C \rho^{N-2} \delta + (1-\lambda)^{N-2} \int_{B_\rho} |Dv|^2$$

or

$$\int_{B_\rho} |Du(k)|^2 \leq C \delta + \int_{B_\rho} |Du|^2.$$

This implies that

$$\int_{B_\rho} |Du|^2 \leq \liminf_{k \rightarrow \infty} \int_{B_\rho} |Du(k)|^2 \leq C \delta + \int_{B_\rho} |Dv|^2.$$

$\delta > 0$ had been arbitrary small, so u is minimizing on each $B_{r_1} \subset B_\rho \subset B_1$. Choose $u = v$ to deduce the strong convergence of energy, (ii). \square

3.13. ϵ -HÖLDER REGULARITY LEMMA

In this section we are going to prove an ϵ -regularity lemma for the Hölder continuity of energy minimizing maps in the spirit of the Schoen-Uhlenbeck regularity theorem.

Lemma 3.13.1. *Let $u(k) \in W^{1,2}(B_1, \mathcal{A}_Q(\mathcal{N}))$ be a sequence of energy minimizers with*

$$\lim_{k \rightarrow \infty} E(u(k), B_1) = 0.$$

For a subsequence, not relabeled we can find $a_l \in W^{1,2}(B_1, \mathcal{A}_{Q_l}(\mathbb{R}^m))$, $\sum_{l=1}^L Q_l = Q$, a sequence of points $p_l(k) \in \mathcal{N}$ s.t.

(i) a_l is Dirichlet minimizing,

(ii) $\mathcal{G}(\sigma_k^{-1}u(k), a(k)) \rightarrow 0$ in $L^1(B_1)$ for $\sigma_k^2 = E(u(k), B_1)$, $a(k) = \sum_{l=1}^L a_l \oplus \sigma_k^{-1}p_l(k)$

(iii) $\lim_{k \rightarrow \infty} \sigma_k^{-2} E(u(k), B_R) = \sum_{l=1}^L \int_{B_R} |Da_l|^2$ for any $0 < R < 1$.

Proof. To every $u(k)$ fix a "mean" $T(k) \in \mathcal{A}_Q(\mathbb{R}^m)$. We may assume that $T(k) \in \mathcal{A}_Q(\mathcal{N})$, if not replace it by $T \in \mathcal{A}_Q(\mathcal{N})$ with $\mathcal{G}(T(k), T) = \min_{S \in \mathcal{A}_Q(\mathcal{N})} \mathcal{G}(T(k), S)$. T has still the property of a "mean", as one may check:

$$\begin{aligned} \int_{B_1} \mathcal{G}(u(k), T)^2 &\leq 2 \int_{B_1} \mathcal{G}(u(k), T(k))^2 + 2 \int_{B_1} \mathcal{G}(T(k), T)^2 \\ &\leq 4 \int_{B_1} \mathcal{G}(u(k), T(k))^2 \leq 4C \int_{B_1} |Du(k)|^2. \end{aligned}$$

Next we apply the concentration compactness lemma, A.1, to the sequence of tuples $\sigma_k^{-1}u(k), \sigma_k^{-1}T(k)$. For a subsequence not relabeled, we get maps $a_l \in W^{1,2}(\mathcal{A}_{Q_l}(B_1), \mathbb{R}^m)$, a related sequence $t_l(k) = \sigma_k^{-1}p_l(k) \in \text{spt}(\sigma_k^{-1}T(k))$. (ii) is a consequence of the concentration compactness lemma. It remains to prove that the a_l 's are Dirichlet minimizing and that the strong convergence of energy (iii) holds. The concentration compactness lemma implies $\sum_{l=1}^L \int_{B_R} |Da_l|^2 \leq \liminf_{k \rightarrow \infty} \sigma_k^{-2} E(u(k), B_R)$ for all $0 < R < 1$ as a consequence of the lower semi-continuity of energy.

ψ_k denotes the 1-Lipschitz retraction map onto $B_{\sigma_k^{-\frac{1}{2}}} \subset \mathbb{R}^m$ defined as

$$\psi_k(z) = \begin{cases} z, & \text{for } |z| < \sigma_k^{-\frac{1}{2}} \\ \sigma_k^{-\frac{1}{2}} \frac{z}{|z|}, & \text{for } |z| \geq \sigma_k^{-\frac{1}{2}} \end{cases}$$

Furthermore we set

$$(3.13.1) \quad a(k) = \sum_{l=1}^L a_l \oplus \sigma_k^{-1}p_l(k) \text{ and } \tilde{a}(k) = \sum_{l=1}^L \psi_k(a_l) \oplus \sigma_k^{-1}p_l(k).$$

We still have $\mathcal{G}(\tilde{a}(k), \sigma_k^{-1}u(k)) \rightarrow 0$ in $L^2(B_1)$ because

$$\begin{aligned} \int_{B_1} \mathcal{G}(\tilde{a}(k), \sigma_k^{-1}u(k))^2 &\leq 2 \int_{B_1} \mathcal{G}(a(k), \sigma_k^{-1}u(k))^2 + 2 \int_{B_1} \mathcal{G}(a(k), \tilde{a}(k))^2 \\ &\leq 2 \int_{B_1} \mathcal{G}(a(k), \sigma_k^{-1}u(k))^2 + 2 \sum_{l=1}^L \int_{B_1 \cap \{x: |a_l|(x) \geq \sigma_k^{-\frac{1}{2}}\}} |a_l|^2. \end{aligned}$$

$\int_{B_1} \mathcal{G}(a(k), \sigma_k^{-1}u(k))^2 \rightarrow 0$ in $L^2(B_1)$ due to the concentration compactness lemma, the second integral tends to 0 as $k \rightarrow \infty$ (since $\sigma_k \rightarrow 0$) and $|a_l| \in L^2(B_1)$.

Let $\frac{1}{2} < R < 1$ be fixed and $0 < \delta < 1 - R$ be arbitrary small but fixed as well. Choose $0 < \epsilon < \lambda < \frac{\delta}{3}$ s.t.

$$C \frac{2^{N+2}}{1-R} \lambda < \delta \quad \text{and} \quad C \frac{2^{N+3}}{\lambda^{N-2}(1-R)} \epsilon < \frac{1}{2} d^2,$$

where C is the constant of the Luckhaus' lemma 3.11.1 and $d > 0$ the size of the tubular neighbourhood to \mathcal{N} .

Using the classical notation for rescaling a function $f_r(x) = f(rx)$, Fatou's lemma states

$$\begin{aligned} & \int_{1-R}^1 \liminf_{k \rightarrow \infty} \int_{\mathcal{S}^{N-1}} \left(\frac{|Du(k)_r|^2}{\sigma_k^2} + |Da(k)_r|^2 + \frac{\mathcal{G}(\tilde{a}(k)_r, \sigma_k^{-1}u(k)_r)^2}{\epsilon^2} \right) \\ & \leq \liminf_{k \rightarrow \infty} \int_{1-R}^1 r^{-N} \int_{\partial B_r} \left(r^2 \left(\frac{|Du(k)_r|^2}{\sigma_k^2} + |Da(k)_r|^2 \right) + \frac{\mathcal{G}(\tilde{a}(k)_r, \sigma_k^{-1}u(k)_r)^2}{\epsilon^2} \right) \\ & \leq 2^N \liminf_{k \rightarrow \infty} \int_{B_1 \setminus B_{1-R}} \left(\frac{|Du(k)_r|^2}{\sigma_k^2} + |Da(k)_r|^2 + \frac{\mathcal{G}(\tilde{a}(k)_r, \sigma_k^{-1}u(k)_r)^2}{\epsilon^2} \right) \leq 2^{N-1}. \end{aligned}$$

Hence there must be a radius $R < \rho < 1$ and a subsequence not relabelled with

$$\int_{\mathcal{S}^{N-1}} \left(\frac{|Du(k)_\rho|^2}{\sigma_k^2} + |Da(k)_\rho|^2 + \frac{\mathcal{G}(\tilde{a}(k)_\rho, \sigma_k^{-1}u(k)_\rho)^2}{\epsilon^2} \right) < \frac{2^{N+2}}{1-R}.$$

As in the proof of the compactness of minimizers, lemma 3.12.1, apply Luckhaus' lemma 3.11.1 to each tuple $\sigma_k^{-1}u(k)_\rho, \tilde{a}(k)_\rho$ to get $\tilde{\varphi}(k) \in W^{1,2}(B_1 \setminus B_{1-\lambda}, \mathcal{A}_Q(\mathbb{R}^m))$ with the properties that

- (i) $\tilde{\varphi}(k)(x) = \sigma_k^{-1}u(k)_\rho(x)$ for $|x| = 1$ and $\tilde{\varphi}(k)(x) = \tilde{a}(k)_\rho(\frac{x}{1-\lambda})$ for $|x| = 1-\lambda$;
- (ii) $\int_{B_1 \setminus B_{1-\lambda}} |D\tilde{\varphi}(k)|^2 \leq C \frac{2^{N+2}}{1-R} \lambda < \delta$;
- (iii) for some $a^2 \leq C \frac{2^{N+3}}{\lambda^{N-2}(1-R)} \epsilon < \frac{1}{2} d^2$

$$\tilde{\varphi}(k)(x) \in \{z \in \mathbb{R}^m : \text{dist}(z, \sigma_k^{-1}u(k)(\partial B_\rho) \cup \tilde{a}(k)(\partial B_\rho)) < a\}.$$

Due to (3.13.1) we have $\text{dist}(\sigma_k^{-1}\tilde{a}(k)(x), \mathcal{A}_Q(\mathcal{N}))^2 \leq \sum_{l=1}^L \sigma_k^2 |\psi_k(a_l)(x)|^2 \leq L\sigma_k$, so that $\text{dist}(\sigma_k\tilde{\varphi}(k)(x), \mathcal{A}_Q(\mathcal{N})) < a + \sqrt{L}\sigma_k$. Hence it is in the tubular neighborhood and

$$\varphi(k)(x) = \sigma_k^{-1} \Pi \left(\sigma_k \tilde{\varphi}(k) \left(\frac{x}{\rho} \right) \right)$$

is well-defined. Furthermore it satisfies

- (i) $\varphi(k)(x) = \sigma_k^{-1}u(k)(x)$ for $|x| = \rho$, $\varphi(k)(x) = \sigma_k^{-1} \Pi(\sigma_k \tilde{a}(k)(\frac{x}{1-\lambda}))$ for $|x| = (1-\lambda)\rho$;
- (ii) $\int_{B_\rho \setminus B_{(1-\lambda)\rho}} |D\varphi(k)|^2 \leq C\rho^{N-2}\delta$.

Given competitors $c_l \in W^{1,2}(B_\rho, \mathcal{A}_Q(\mathbb{R}^m))$ to each a_l . As in (3.13.1) set

$$c(k) = \sum_{l=1}^L c_l \oplus \sigma_k^{-1}p_l(k) \text{ and } \tilde{c}(k) = \sum_{l=1}^L \psi_k(a_l) \oplus \sigma_k^{-1}p_l(k).$$

As for $\tilde{a}(k)$ we have $\text{dist}(\sigma_k\tilde{c}(k)(x), \mathcal{A}_Q(\mathcal{N})) \leq L\sigma_k$. Remark 3.10.1 (iv) states

$$(1 - 2\sqrt{L\sigma_k}A) |D(\Pi(\sigma_k\tilde{c}(k)))|^2 \leq \sigma_k^2 |D\tilde{c}(k)|^2 \leq \sigma_k^2 |Dc(k)|^2$$

with $A = \sup_{p \in \mathcal{N}} \|A_p\|_\infty$ and $|Dc(k)|^2 = \sum_{l=1}^L |D\psi_k(u_l)|^2 \leq \sum_{l=1}^L |Du_l|^2$ because ψ_k is a 1-Lipschitz retraction. We can define a competitor to $u(k)$ by

$$v(k)(x) = \begin{cases} u(k)(x), & \text{for } \rho < |x| \leq 1 \\ \sigma_k \varphi(k)(x), & \text{for } (1-\lambda)\rho < |x| \leq \rho \\ \Pi(\sigma_k \tilde{c}(k)(\frac{x}{1-\lambda})), & \text{for } |x| \leq (1-\lambda)\rho. \end{cases}$$

Hence by minimality of each $u(k)$ we get

$$\int_{B_1} |Du(k)|^2 \leq \int_{B_1} |Dv(k)|^2 \leq \int_{B_1 \setminus B_\rho} |Du(k)|^2 + C\sigma_k^2\delta + \frac{\sigma_k^2(1-\lambda)^{N-2}}{1-2\sqrt{L\sigma_k}A} \int_{B_\rho} |Dc(k)|^2$$

or

$$\sigma_k^{-2} \int_{B_\rho} |Du(k)|^2 \leq C\delta + \frac{1}{1 - 2\sqrt{L}\sigma_k A} \sum_{l=1}^L \int_{B_\rho} |Dc_l|^2.$$

This implies that, by lower semicontinuity of the energy,

$$(3.13.2) \quad \sum_{l=1}^L \int_{B_\rho} |Da_l|^2 \leq \liminf_{k \rightarrow \infty} \sigma_k^{-2} \int_{B_\rho} |Du(k)|^2 \leq C\delta + \sum_{l=1}^L \int_{B_\rho} |Dc_l|^2.$$

$\delta > 0$ can be taken arbitrary small, so each a_l must be minimizing on $B_R \subset B_\rho \subset B_1$. Choose $c_l = a_l$ for each l in (3.13.2) to deduce the strong convergence of energy, i.e. (iii). \square

Lemma 3.13.2. *There exists $\epsilon_0 > 0$ and $\alpha > 0, C > 1$ depending on N, Q, \mathcal{N} with the property that, if $u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathcal{N}))$ is energy minimizing with*

$$(3.13.3) \quad E(u, B_{R_0}(y_0)) \leq \epsilon_0 \text{ for some } B_{R_0}(y_0) \subset \Omega,$$

then $|Du|$ is an element of the Morrey space $L^{2,N-2+2\alpha}(B_{\frac{R_0}{2}}(y_0))$. More precisely we have the estimate

$$(3.13.4) \quad E(u, B_r(y)) \leq C \left(\frac{r}{R} \right)^{2\alpha} E(u, B_R(y)) \forall y \in \overline{B_{\frac{R_0}{2}}(y_0)}, 0 < r \leq R \leq \frac{R_0}{2}.$$

Furthermore $u \in C^{0,\alpha}(B_{\frac{R_0}{2}}(y_0))$.

Proof. First we will prove the following statement and show thereafter how it implies (3.13.4).

$\exists \epsilon_1 > 0, 0 < \gamma < 1$ depending on N, Q, \mathcal{N} s.t. if $u \in W^{1,2}(B_R(y), \mathcal{A}_Q(\mathcal{N}))$ is energy minimizing and $E(u, B_R(y)) < \epsilon_1$ then

$$(3.13.5) \quad E(u, B_{\frac{R}{2}}(y)) < \gamma E(u, B_R(y)).$$

Indeed, fix $\gamma < 2^{-2\alpha_0}$, where $\alpha_0 = \alpha_0(N, Q) > 0$ is the Hölder exponent for Dirichlet minimizers into \mathbb{R}^m , compare [12, Theorem 0.9]. Suppose such an $\epsilon_1 > 0$ does not exist, hence there are $v(k) \in W^{1,2}(B_{R_k}(y_k), \mathcal{A}_Q(\mathcal{N}))$ energy minimizing failing (3.13.5), i.e. $E(v(k), B_{\frac{R_k}{2}}(y_k)) \geq \gamma E(v(k), B_{R_k}(y_k))$ and $\sigma_k^2 = E(v(k), B_{R_k}(y_k)) \rightarrow 0$ as $k \rightarrow \infty$. Consider the rescaled sequence

$$u(k)(x) = v(k)(y_k + R_k x) \text{ i.e. } E(u(k), B_1) = E(v(k), B_{R_k}(y_k)) = \sigma_k^2.$$

So we can apply the previous lemma 3.13.2: for a subsequence $u(k)$, not relabeled, there are Dirichlet minimizing $a_l \in W^{1,2}(B_1, \mathcal{A}_{Q_l}(\mathbb{R}^m))$ ($\sum_{l=1}^L Q_l = Q$) and a sequence of points $p_l(k) \in \mathcal{N}$ such that for $a(k) = \sum_{l=1}^L a_l \oplus \sigma_k^{-1} p_l(k)$ one has

$$(i) \quad \mathcal{G}(\sigma_k^{-1} u(k), a(k)) \rightarrow 0 \text{ in } L^2(B_1);$$

$$(ii) \quad \lim_{k \rightarrow \infty} \sigma_k^{-2} E(u(k), B_R) = \sum_{l=1}^L E(a_l, B_R) \text{ for all } 0 < R < 1.$$

We firstly observe that this implies $\sum_{l=1}^L E(a_l, B_{\frac{1}{2}}) \geq \gamma$ because

$$\sigma_k^{-2} E(u(k), B_{\frac{1}{2}}) = \sigma_k^{-2} E(v(k), B_{\frac{R_k}{2}}(y_k)) \geq \gamma.$$

Secondly

$$\sum_{l=1}^L E(a_l, B_{\frac{1}{2}}) = \lim_{k \rightarrow \infty} \sigma_k^{-2} E(u(k), B_{\frac{1}{2}}) \geq \liminf_{k \rightarrow \infty} \gamma \sigma_k^{-2} E(u(k), B_1) \geq \sum_{l=1}^L E(a_l, B_1).$$

So there must be a nontrivial a_l , with $E(a_l, B_{\frac{1}{2}}) \geq \gamma E(a_l, B_1)$. But a_l is Dirichlet minimizing and therefore $E(a_l, B_{\frac{1}{2}}) \leq 2^{-2\alpha_0} E(a_l, B_1)$. This is a contradiction. Set $\epsilon_0 = 2^{-N} \epsilon_1 > 0$, then (3.13.4) holds because, if $E(u, B_{R_0}(y_0)) < \epsilon_0$, then

$$E(u, B_R(y)) \leq E(u, B_{\frac{R_0}{2}}(y)) \leq 2^N E(u, B_{R_0}(y_0)) \quad \forall y \in \overline{B_{\frac{R_0}{2}}(y_0)}, 0 < R < \frac{R_0}{2},$$

as a consequence of the monotonicity formula (3.10.9).

Induction on (3.13.5) gives $E(u, B_{2^{-k}R}(y)) \leq \gamma^k E(u, B_R(y))$ for all $k \in \mathbb{N}$ and any $y \in \overline{B_{\frac{R_0}{2}}(y_0)}$, $0 < R < \frac{R_0}{2}$. Choose $k \in \mathbb{N}$ s.t. $2^{-k-1}R < r \leq 2^{-k}R$ for $r < R$. Then by monotonicity (3.10.9) and the estimates above we have

$$E(u, B_r(y)) \leq E(u, B_{2^{-k}R}(y)) \leq \frac{1}{\gamma} \gamma^{k+1} E(u, B_R(y)) \leq \frac{1}{\gamma} \left(\frac{r}{R} \right)^{2\alpha} E(u, B_R(y))$$

for $2\alpha = \frac{-\ln(\gamma)}{\ln(2)}$.

(3.13.5) implies that $|Du|$ is an element of the Morrey space $L^{2,N-2+2\alpha}(B_{\frac{R_0}{2}}(y_0))$. The Hölder continuity then follows classically. \square

3.14. PROPERTIES OF THE SINGULAR SET $\text{sing}_H u$

In this section let $u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathcal{N}))$ be a fixed energy minimizing map. For any $B_{R_0}(y) \subset \Omega$, the monotonicity formula, (3.10.9), gives

$$\Theta_u(y) = \inf_{0 < R \leq R_0} E(u, B_R(y)) \leq E(u, B_R(y)) \leq E(u, B_{R_0}(y)) \quad \forall 0 < R \leq R_0.$$

For any sequence $R_k \rightarrow 0$ and $y \in \Omega$ we may consider the rescaled sequence $v(k)(x) = u_{y,R_k}(x) = u(y + R_k x)$ and observe that for any $r > 0$, sufficient large $k \in \mathbb{N}$, i.e. $R_k \leq \frac{R_0}{r}$

$$E(v(k), B_r) = E(u, B_{rR_k}(y)) \leq E(u, B_{R_0}(y)).$$

The compactness result, lemma 3.12.1, asserts for a subsequence $v(k')$ there is $\varphi \in W^{1,2}(\mathbb{R}^N, \mathcal{A}_Q(\mathcal{N}))$ energy minimizing with $\mathcal{G}(v(k), \varphi) \rightarrow 0$ in $L^2_{loc}(\mathbb{R}^N)$ and

$$(3.14.1) \quad E(\varphi, B_r) = \lim_{k \rightarrow \infty} E(v(k), B_r) = \lim_{k \rightarrow \infty} E(u, B_{rR_k}(y)) = \Theta_u(y) \quad \forall R > 0.$$

Furthermore the monotonicity formula, (3.10.9) gives

$$0 = \int_{B_R \setminus B_r} |x|^{2-N} \left| \frac{\partial \varphi}{\partial r} \right|^2 \quad \forall 0 < r < R.$$

So, $0 = \left| \frac{\partial \varphi}{\partial r} \right|^2 = \sum_{l=1}^Q \left| \frac{\partial \varphi_l}{\partial r} \right|^2 = 0$ a.e.. Integrating this in r gives $\varphi(\lambda x) = \varphi(x)$ for all $\lambda > 0$ and $x \in \mathbb{R}^N$. This homogeneous degree zero property is characteristic for tangent maps, hence we define classically:

Definition 3.14.1. A zero homogenous function $\varphi \in W^{1,2}(\mathbb{R}^N, \mathcal{A}_Q(\mathcal{N}))$ is called tangent map to u at $y \in \Omega$ if

$$\exists R_k \rightarrow 0 \text{ with } \mathcal{G}(u_{y,R_k}, \varphi) \rightarrow 0 \text{ in } L^2_{loc}(\mathbb{R}^N).$$

3.14.1. Properties of homogeneous degree zero minimizers. Let us consider $\varphi \in W^{1,2}(\mathbb{R}^N, \mathcal{A}_Q(\mathcal{N}))$ be energy minimizing and zero homogeneous, i.e. $\varphi(\lambda x) = \varphi(x)$ for all $x \in \mathbb{R}^N, \lambda > 0$. Every tangent map, definition 3.14.1, has this property. In this section we state some consequences. First of all one observe that the multivalued case does not differ from the single valued, "classical" case. Our presentation follows very closely L.Simon's in [21, section 3]. The analysis of tangent maps enables a stratification procedure, section 3.14.2. It is a direct modification of a result

by F. Almgren, [2]. As a consequence we will be able to get an estimate on the singular set $\text{sing}_H u$.

$$(3.14.2) \quad \Theta_\varphi(y) \text{ takes its maximum in } y = 0.$$

Indeed, fix $y \in \mathbb{R}^N$, for any $0 < R$ combining the monotonicity (3.10.11) with $E(\varphi, B_r(0)) = \Theta_\varphi(0) \forall r > 0$ gives

$$\begin{aligned} 2 \int_{B_R(y)} |x - y|^{2-N} \left| \frac{\partial \varphi}{\partial r_y} \right|^2 + \Theta_\varphi(y) &= E(\varphi, B_R(y)) \\ &\leq \left(1 + \frac{|y|}{R}\right)^{N-2} E(\varphi, B_{R+|y|}(0)) = \left(1 + \frac{|y|}{R}\right)^{N-2} \Theta_u(0); \end{aligned}$$

with $\frac{\partial}{\partial r_y}$ we want to emphasize the center y i.e. it is the directional derivative in the radial direction $\frac{x-y}{|x-y|}$. Taking the limit $R \rightarrow \infty$ we get

$$(3.14.3) \quad 2 \int_{\mathbb{R}^N} |x - y|^{2-N} \left| \frac{\partial \varphi}{\partial r_y} \right|^2 + \Theta_\varphi(y) \leq \Theta_u(0) = \Theta_\varphi(0).$$

Definition 3.14.2. Let $\varphi \in W^{1,2}(\mathbb{R}^m, \mathcal{A}_Q(N))$ be a homogeneous degree 0 energy minimizer. Then we define

$$S(\varphi) = \{y \in \mathbb{R}^N : \Theta_\varphi(y) = \Theta_\varphi(0)\}.$$

We next claim that

$$(3.14.4) \quad S(\varphi) \text{ is a linear subspace of } \mathbb{R}^N$$

$$(3.14.5) \quad \text{and } \varphi(x+y) = \varphi(x) \text{ for all } x \in \mathbb{R}^N, y \in S(\varphi)$$

To show (3.14.4) and (3.14.5) observe that for $y \in S(\varphi)$, equality in (3.14.3) implies $\frac{\partial \varphi}{\partial r_y} = 0$ i.e.

$$\varphi(y + \lambda x) = \varphi(y + x) \quad \forall x \in \mathbb{R}^N, \lambda > 0$$

Combing this with, $\varphi(\tilde{\lambda}x) = \varphi(x) \quad \forall x \in \mathbb{R}^N, \tilde{\lambda} > 0$ gives

$$\begin{aligned} \varphi(x) &= \varphi(\lambda x) \\ &= \varphi(y + (\lambda x - y)) = \varphi(y + \lambda^{-2}(\lambda x - y)) = \varphi(\lambda^{-1}x + (y - \lambda^{-2}y)) \\ &= \varphi(x + (\lambda - \lambda^{-1})y) = \varphi(x + \mu y) \end{aligned}$$

where $\mu = \lambda - \lambda^{-1}$ is an arbitrary real number. This implies naturally $E(u, B_R(0)) = E(u, B_R(\mu y))$ and $\Theta_\varphi(0) = \Theta_\varphi(\mu y)$ for all $\mu \in \mathbb{R}$ and $y \in S(\varphi)$.

3.14.2. Consequences for $\text{sing}_H u$. The obtained results gives us equivalent identifications of the Hölder regular set.

Lemma 3.14.1. *Let $u \in W^{1,2}(\Omega, \mathcal{A}_Q(N))$ be energy minimizing, then the following are equivalent*

- (i) $y \in \text{reg}_H u$;
- (ii) $\Theta_u(y) = 0$;
- (iii) u has a constant tangent map φ at y ;
- (iv) $\dim S(\varphi) = N$ for some tangent map φ of u at y .

Proof. (i) \Rightarrow (iii): Let φ be any tangent map of u at y . Passing to a subsequence we have $u_{y, R_k}(x) = u(y + R_k x)$ converging locally a.e. to φ . Hence for a.e. x, x' we have

$$\mathcal{G}(\varphi(x), \varphi(x')) = \lim_{k \rightarrow \infty} \mathcal{G}(u(y + R_k x), u(y + R_k x')) \leq \liminf_{k \rightarrow \infty} C R_k^\alpha |x - x'| = 0.$$

Thus $\varphi \equiv \text{const.}$.

(ii) \Leftrightarrow (iii) : This equivalence is obvious.

(iii) \Leftrightarrow (iv) : This equivalence just follows by definition and the last observation in the previous section.

(ii) \Rightarrow (i) : If $\Theta_u(y) = 0$ there is a $R > 0$ s.t. $E(u, B_R(y)) < \epsilon_0$, where $\epsilon_0 > 0$ is the constant of lemma 3.13.1. Then this lemma states $u \in C^{0,\alpha}(B_{\frac{R}{2}}(y))$ and so $y \in \text{reg}_H u$. \square

Remark 3.14.3. For single valued, "classical" harmonic functions, lemma 3.14.1 implies

$$\text{reg } u = \text{reg}_H u \text{ and so } \text{sing } u = \text{sing}_H u$$

Furthermore lemma 3.14.1 has the following simple consequences as in the single valued setting.

Lemma 3.14.2. $\mathcal{H}^{N-2}(\text{sing}_H u) = 0$

Proof. This is a classical consequence of $|Du|^2$ being in L^1 and $\text{sing}_H u = \{y : \Theta_u(y) > \epsilon_0\}$. \square

One defines

$$(3.14.6) \quad S_j = \{y \in \text{sing}_H u : \dim S(\varphi) \leq j \text{ for all tangent maps } \varphi \text{ at } y\}.$$

We first observe that

$$(3.14.7) \quad \text{sing}_H u = S_{N-1} = S_{N-2} = S_{N-3}.$$

Indeed, suppose not. Then there would be a tangent map φ , which is a non constant homogenous degree zero minimizer with $N-1 \geq \dim S(\varphi) \geq N-2$. This contradicts lemma 3.14.2 because $S(\varphi) \subset \text{sing}_H \varphi$ and

$$+\infty = \mathcal{H}^{N-2}(S(\varphi)) = \mathcal{H}^{N-2}(\text{sing}_H u).$$

As L. Simon mentions in [21, section 3.4] one notice:

"The subsets S_j are mainly important because of the following lemma, which is a direct modification of the corresponding result for minimal surfaces by F. Almgren [2]; the lemma can be thought of as a refinement of the "dimension reducing" argument of Federer [6] (for this see also the discussion in the appendix of [20])." ³

Classically a characterization of S_j implies a δ -approximation property which then itself implies the following two results. Their classical proofs can be found in [21, section 3.4, Lemma 1 & Corollary 1]

Lemma 3.14.3. *For each $j = 0, \dots, N-3$, $\dim S_j \leq j$, and for each $t > 0$, $S_0 \cap \{y : \Theta_u(y) = t\}$ is a discrete set.*

Corollary 3.14.4. *$\dim \text{sing}_H u \leq N-3$. More generally, if all tangent maps φ of u satisfy $\dim S(\varphi) \leq j_0 \leq N-3$ then $\dim \text{sing}_H u \leq j_0$.*

This corollary clearly shows theorem 3.8.1.

³L. Simon, [21], page 54

APPENDIX A. THE LUCKHAUS LEMMA

A classical result due to S. Luckhaus is concerned with the extension of a map that is defined on the boundary of an annulus $\partial(B_1 \setminus B_{1-\lambda})$ into the interior. Its proof for single valued functions is nowadays classical and can be found for instance in [17]. We mentioned the result already in section 3.11. We want to give now a complete intrinsic proof for Q -valued functions. Our formulation is based on S. Luckhaus' original, [16, Lemma 1] and the one of R. Mosers, [17, Lemma 4.4].

Lemma A.1. *There is a constants $C, C_\infty > 0$ depending only on the dimension N such that the following holds:*

Suppose $\lambda = \frac{1}{L}$, $\epsilon = \frac{1}{lL} \leq \lambda$, $l, L \in \mathbb{N}, L > 2$ given, furthermore let $u, v \in W^{1,2}(\mathcal{S}^{N-1}, \mathcal{A}_Q(\mathbb{R}^m))$ with

$$(A.1) \quad \int_{\mathcal{S}^{N-1}} |D_\tau u|^2 + |D_\tau v|^2 + \frac{\mathcal{G}(u, v)^2}{\epsilon^2} = K^2;$$

then there exists $\varphi \in W^{1,2}(B_1 \setminus B_{1-\lambda}, \mathcal{A}_Q(\mathbb{R}^m))$ with the following properties

$$(A.2) \quad \varphi(x) = \begin{cases} u(x), & \text{if } |x| = 1 \\ v(\frac{x}{1-\lambda}), & \text{if } |x| = 1 - \lambda \end{cases}$$

$$(A.3) \quad \int_{B_1 \setminus B_{1-\lambda}} |D\varphi|^2 \leq C Q \lambda K^2$$

$$(A.4) \quad \varphi(x) \in \{y \in \mathbb{R}^m : \text{dist}(y, u(\mathcal{S}^{N-1}) \cup v(\mathcal{S}^{N-1})) < a\}$$

for some $a > 0$ with $a^2 \leq C_\infty Q^2 \lambda^{2-N} \epsilon K^2$.

Remark A.1. The L^∞ -bound, (A.4), is a little bit weaker then tho stated in Lemma 3.11.1. The dependence of the constants on N, m and Q is more precise.

The proof of Lemma A.1 is very close to S. Luckhaus orginial one, nicely presented by R. Moser, [17, Lemma 4.4]. It splits in 3 parts:

- (1) a decomposition \mathcal{G} of the sphere \mathcal{S}^{N-1} that is bilipschitz to cubical decomposition of $\partial[-1, 1]^N$ into parallel disjoint cubes of side length λ . This is a measure theoretic argument;
- (2) two types of extensions on cubes;
- (3) a recursive definition of ϕ on cubical subsets $F \times [0, \lambda]$ where F is a k -dimensional face int the cubical decomposition. It always takes advantage that ϕ had already be defind on all $F' \times [0, \lambda]$ for lower dimensional faces F' .

Studying S. Luckhaus' original proof one notice that only for the extensions on $F \times [0, \lambda]$, F being a 1-dimensional face, the linear structure of $W^{1,2}(F, \mathbb{R}^m)$ is needed. C. De Lellis presented a possible replacement $W^{1,2}(F, \mathcal{A}_Q(\mathbb{R}^m))$ in [13]. His version does not preserve the L^∞ -bound, (A.4), compare remark below. Nonetheless following his ideas one can recover the bound, lemma. Our proof does not contain essentially new ideas. It boils down to replacing lemma E.2 in the proof proposed by C. De Lellis, [13] or the linear extension in S. Luckhaus original one by lemma. Nonetheless we decided to give a complete detailed proof.

As mentioned, in part 1 one uses the bilipschitz equivalence between B_1 and $[-1, 1]^N$ and their boundaries \mathcal{S}^{N-1} and $\partial[-1, 1]^N$. Therefore we list in the following remark some terminology and constants appearing in this context. Since only the extensions, part 2, differ slightly from the already existing proofs they are presented first thereafter. Finally we will proceed with part 1 and 3.

Remark A.2. $|x|^2 = |x|_2^2 = (x_1)^2 + \dots + (x_n)^2$ denotes the Euclidean norm on \mathbb{R}^n and $|x|_\infty = \max\{|x_1|, \dots, |x_n|\}$ the supremum norm. Let $B_1 = \{|x|_2 < 1\}$ be the unite ball and $[-1, 1]^n = \{|x|_\infty < 1\}$ the standard cube in \mathbb{R}^n . Set $H(x) = \frac{|x|_\infty}{|x|_2}x$ and $G(x) = \frac{|x|_2}{|x|_\infty}x$, then $H = G^{-1}$ and $H : [-1, 1]^n \rightarrow B_1$ so their boundaries $H : \partial[-1, 1]^n \rightarrow \mathcal{S}^n$. δ_{ij} denote the Euclidean metric on \mathbb{R}^n or the pullback metric for a submanifold in \mathbb{R}^n . Furthermore let $g = G^\sharp \delta$ and $h = H^\sharp \delta$ be the pullback metrics on $B_1, [-1, 1]^n$ respectively. One calculates

$$\det(g) = \left(\frac{|x|_2}{|x|_\infty} \right)^{2n} = \det(g|_{\mathcal{S}^{n-1}}).$$

Furthermore the spectrum of g^{-1} is contained in $[1 - \left(\frac{|x|_\infty}{|x|_2}\right)^2, 1 + \left(\frac{|x|_\infty}{|x|_2}\right)^2]$. The eigenvalues of $g|_{\mathcal{S}^{n-1}}$ are $\left(\frac{|x|_2}{|x|_\infty}\right)^4$ and $n-2$ times $\left(\frac{|x|_2}{|x|_\infty}\right)^2$. For h we therefore have

$$\det(h) = \left(\frac{|x|_\infty}{|x|_2} \right)^{2n} = \det(h|_{\partial[-1, 1]^n}).$$

The spectrum of h^{-1} is contained in $[\left(\frac{|x|_2}{|x|_\infty}\right)^4 - \left(\frac{|x|_2}{|x|_\infty}\right)^2, \left(\frac{|x|_2}{|x|_\infty}\right)^4 + \left(\frac{|x|_2}{|x|_\infty}\right)^2]$. The eigenvalues of $h|_{\partial[-1, 1]^n}$ are $\left(\frac{|x|_\infty}{|x|_2}\right)^4$ and $n-2$ times $\left(\frac{|x|_\infty}{|x|_2}\right)^2$. This has for instance the following implications:

(A.5)

$$\begin{aligned} \int_{B_1} |D\varphi|^2 &= \int_{[-1, 1]^n} h^{ij} \frac{\partial \phi}{\partial x^i} \frac{\partial \phi}{\partial x^j} \sqrt{\det(h)} \leq 3 \int_{[-1, 1]^n} |D\phi|^2 \text{ for } \varphi = \phi \circ G \\ \int_{\mathcal{S}^{n-1}} |D_\tau \varphi|^2 &= \int_{\partial[-1, 1]^n} h^{ij} \frac{\partial \phi}{\partial x^i} \frac{\partial \phi}{\partial x^j} \sqrt{\det(h|_{\partial[-1, 1]^n})} \leq c_n \int_{\partial[-1, 1]^n} |D_\tau \phi|^2, \end{aligned}$$

since $\left\{ \left(\frac{|x|_2}{|x|_\infty}\right)^4 + \left(\frac{|x|_2}{|x|_\infty}\right)^2 \right\} \left(\frac{|x|_\infty}{|x|_2}\right)^n \leq 3 \forall n$ and $\left(\frac{|x|_2}{|x|_\infty}\right)^{4-n} \leq c_n$ for $c_2 = 2, c_3 = \sqrt{3}$ and $c_n = 1$ for $n \geq 4$. Similarly one calculates for $\phi = \varphi \circ H$

$$\begin{aligned} \int_{[-1, 1]^n} |D\phi|^2 &= \int_{B_1} g^{ij} \frac{\partial \varphi}{\partial x^i} \frac{\partial \varphi}{\partial x^j} \sqrt{\det(g)} \leq n^{\frac{n}{2}} (1 + n^{-1}) \int_{B_1} |D\varphi|^2 \\ \int_{\partial[-1, 1]^n} |D_\tau \phi|^2 &= \int_{\mathcal{S}^{n-1}} g^{ij} \frac{\partial \varphi}{\partial x^i} \frac{\partial \varphi}{\partial x^j} \sqrt{\det(g|_{\mathcal{S}^{n-1}})} \leq n^{\frac{n-2}{2}} \int_{\mathcal{S}^{n-1}} |D_\tau \varphi|^2. \end{aligned}$$

The extension lemma for faces of dimension $k \geq 3$ is the classical following one:

Lemma A.2. *Given $F = z + [0, \lambda]^n$, $n \geq 3$, a n -dimensional cube of side length λ and $\phi \in W^{1,2}(\partial F, \mathcal{A}_Q(\mathbb{R}^m))$ then there is an extension $\hat{\phi} \in W^{1,2}(F, \mathcal{A}_Q(\mathbb{R}^m))$ with the property that*

$$(A.7) \quad \int_F |D\hat{\phi}|^2 \leq \frac{n}{2(n-2)} \lambda \int_{\partial F} |D_\tau \phi|^2$$

$$(A.8) \quad \hat{\phi}(x) \in \phi(\partial F) \quad \forall x \in F$$

Proof. By a simple scaling argument it is sufficient to prove the lemma for $F = [-1, 1]^n$. Since $n \geq 3$ the 0-homogeneous extension $\hat{\phi}(x) = \phi\left(\frac{x}{|x|_\infty}\right)$ belongs to $W^{1,2}(F, \mathcal{A}_Q(\mathbb{R}^m))$. Direct computations provide the bound (A.5). (A.6) is clearly satisfied. \square

The crucial point is to find a "version" of Lemma A.2 for $n = 2$. The first step is the replacement suggested by C. De Lellis.

Lemma A.3. *Given $F = z + [0, \lambda]^2$, a 2-dimensional cube of side length λ and $\phi \in W^{1,2}(\partial F, \mathcal{A}_Q(\mathbb{R}^m))$ then there is an extension $\hat{\phi} \in W^{1,2}(F, \mathcal{A}_Q(\mathbb{R}^m))$ with the property that*

$$(A.9) \quad \int_F |D\hat{\phi}|^2 \leq 3Q\lambda \int_{\partial F} |D_\tau \phi|^2$$

$$(A.10) \quad \mathcal{G}(\hat{\phi}(x), \hat{\phi}(y))^2 \leq \pi Q^2 \lambda \int_{\partial F} |D_\tau \phi|^2.$$

Proof. By scaling it is sufficient to prove it for $F = [-1, 1]^2$. Furthermore using $\varphi = \phi \circ G, \hat{\phi} = \hat{\varphi} \circ H$ and the estimates (A.5), (A.6) for $n = 2$ we can show the existence of an extension $\hat{\varphi}$ from \mathcal{S}^1 to the disk B_1 , that satisfies

$$(A.11) \quad \int_{B_1} |D\hat{\varphi}|^2 \leq Q \int_{\mathcal{S}^1} |D_\tau \varphi|^2$$

$$(A.12) \quad \mathcal{G}(\hat{\varphi}(x), \hat{\varphi}(y))^2 \leq \pi Q^2 \int_{\mathcal{S}^1} |D_\tau \varphi|^2.$$

The energy bound (A.11) is derived in Proposition 3.10 in [12] as the crucial estimate to establish the optimal Hölder continuity for Dirichlet minimizers in the interior. Although the competitor constructed there satisfies the L^∞ -bound it is not stated. Therefore we present the complete construction. Recall that for a given $\tilde{f} \in W^{1,2}(\mathcal{S}^1, \mathbb{R}^m)$, single valued, there exists a unique harmonic extension $f \in W^{1,2}(B_1, \mathbb{R}^m)$ ($\Delta f = 0$) with $f = \tilde{f}$ on \mathcal{S}^1 and it satisfies

$$(A.13) \quad \int_{B_1} |Df|^2 \leq \int_{\mathcal{S}^1} |D_\tau \tilde{f}|^2$$

and due to the maximum principle for subharmonic functions and 1-dimensional calculus

$$(A.14) \quad |f(x) - f(y)|^2 \leq \sup_{x, y \in \mathcal{S}^1} |\tilde{f}(x) - \tilde{f}(y)|^2 \leq \pi \int_{\mathcal{S}^1} |D_\tau \tilde{f}|^2.$$

Now let be $\varphi \in W^{1,2}(\mathcal{S}^1, \mathcal{A}_Q(\mathbb{R}^m))$ given, as shown in [12, Proposition 1.5] there is an irreducible decomposition $\varphi(x) = \sum_{j=1}^J \sum_{\substack{z \in \mathbb{C} \\ z^{Q_j} = x}} \llbracket \tilde{g}_j(z) \rrbracket$ for all $x \in \mathcal{S}^1$, functions $\tilde{g}_j \in W^{1,2}(\mathcal{S}^1, \mathbb{R}^m)$ and $\sum_{j=1}^J Q_j = Q$. To every \tilde{g}_j let $g_j \in W^{1,2}(B_1, \mathbb{R}^m)$ be the harmonic extension, then set

$$\hat{\varphi}(x) = \sum_{j=1}^J \sum_{\substack{z \in \mathbb{C} \\ z^{Q_j} = x}} \llbracket g_j(z) \rrbracket \text{ for } x \in B_1.$$

Direct computations, compare [12, Lemma 3.12] and (A.13) gives (A.11):

$$\begin{aligned} \int_{B_1} |D\hat{\varphi}|^2 &= \int_{B_1} \sum_{j=1}^J |Dg_j|^2 \leq \sum_{j=1}^J \int_{\mathcal{S}^1} |D_\tau \tilde{g}_j|^2 \\ &\leq Q \sum_{j=1}^J \frac{1}{Q_j} \int_{\mathcal{S}^1} |D_\tau \tilde{g}_j|^2 = Q \int_{\mathcal{S}^1} |D_\tau \varphi|^2. \end{aligned}$$

Furthermore let $x = r \exp(i\alpha)$ then

$$\hat{\varphi}(x) = \sum_{j=1}^J \sum_{l=0}^{Q_j-1} \llbracket g_j \left(r^{\frac{1}{Q_j}} e^{i\frac{\alpha}{Q_j} + il\frac{2\pi}{Q_j}} \right) \rrbracket$$

similar for $y = s \exp(i\beta)$, hence applying (A.14) gives (A.12)

$$\begin{aligned} \mathcal{G}(\widehat{\varphi}(x), \widehat{\varphi}(y))^2 &\leq \sum_{j=1}^J \sum_{l=0}^{Q_j-1} \left| g_j \left(r^{\frac{1}{Q_j}} e^{i \frac{\alpha}{Q_j} + i l \frac{2\pi}{Q_j}} \right) - g_j \left(s^{\frac{1}{Q_j}} e^{i \frac{\beta}{Q_j} + i l \frac{2\pi}{Q_j}} \right) \right|^2 \\ &\leq \pi \sum_{j=1}^J \sum_{l=0}^{Q_j-1} \int_{S^1} |D_\tau \tilde{g}_j|^2 \leq \pi Q^2 \int_{S^1} |D_\tau \varphi|^2. \end{aligned}$$

□

Although $\mathcal{A}_Q(\mathbb{R}^m)$ is not a linear space we will use the following terminology. A map $\phi : [a, b] \rightarrow \mathcal{A}_Q(\mathbb{R}^m)$ is said to be linear, a linear interpolation, between two points $S = \sum_{l=1}^Q \llbracket s_l \rrbracket, T = \sum_{l=1}^Q \llbracket t_l \rrbracket \in \mathcal{A}_Q(\mathbb{R}^m)$ on the interval $[a, b]$ if there exists $\sigma \in \mathcal{P}_Q$ such that

$$\begin{aligned} \mathcal{G}(S, T)^2 &= \sum_{l=1}^Q |s_l - t_{\sigma(l)}|^2 \\ \phi(t) &= \sum_{l=1}^Q \left[\frac{b-t}{b-a} s_l + \frac{t-a}{b-a} t_{\sigma(l)} \right]. \end{aligned}$$

Furthermore one has $\int_a^b |D\phi|^2 = \frac{\mathcal{G}(S, T)^2}{b-a}$ and to any two points $S, T \in \mathcal{A}_Q(\mathbb{R}^m)$ and an interval $[a, b]$ given there exists at least one linear interpolation. (It may not be unique.)

Lemma A.4. *Suppose $\phi \in W^{1,2}(\partial(F \times [0, \lambda]), \mathcal{A}_Q(\mathbb{R}^m))$, $F = [a, b]$ a 1-dimensional face of length $\lambda = b - a$ is given and $\epsilon = \frac{\lambda}{l}, l \in \mathbb{N}$. Furthermore ϕ satisfies the following:*

$t \mapsto \phi(a, t), \phi(b, t)$ are linear between $U(a), V(a)$ and $U(b), V(b)$;

$$\int_F |D_\tau U|^2 + |D_\tau V|^2 + \frac{\mathcal{G}(U, V)^2}{\epsilon^2} = K^2.$$

where $U(x) = \phi(x, 0), V(x) = \phi(x, \lambda) \in W^{1,2}(F, \mathcal{A}_Q(\mathbb{R}^n))$. Then there exists an extension $\widehat{\phi} \in W^{1,2}(F \times [0, \lambda], \mathcal{A}_Q(\mathbb{R}^m))$ satisfying

$$(A.15) \quad \int_{F \times [0, \lambda]} |D\widehat{\phi}|^2 \leq 15Q \lambda K^2;$$

$$(A.16) \quad \text{dist}(\widehat{\phi(x, t)}, U(F) \cup V(F))^2 \leq 5\pi Q^2 \epsilon K^2 \quad \forall (x, t) \in F \times [0, \lambda].$$

Proof. We construct $\widehat{\phi}$ applying the previous extension lemma A.3 several times. Set $a_k = a + k\epsilon$ for $k = 0, \dots, l$, i.e. $a_0 = a, a_l = b$ and for every $k = 1, \dots, l-1$ define $t \mapsto \phi(a_k, t)$ to be a linear interpolation between $U(a_k), V(a_k)$. Pick any $k \in \{0, \dots, l-1\}$ then ϕ is now already defined on $\partial([a_k, a_{k+1}] \times [0, \lambda])$. We may apply lemma A.3 to

$$(x, t) \in [0, \lambda]^2 \mapsto \phi(a_k + \frac{x}{l}, t)$$

and obtain an extension $\phi_k \in W^{1,2}([0, \lambda]^2, \mathcal{A}_Q(\mathbb{R}^m))$. By 1-dimensional calculus one has for $f \in W^{1,2}([c, d], \mathbb{R}) \subset C^{0, \frac{1}{2}}([c, d])$, that

$$\sup_{c \leq x \leq d} f(x)^2 \leq 2|d-c| \int_c^d |f'|^2 + \frac{2}{|d-c|} \int_c^d f^2$$

and therefore

$$\sum_{j=k}^{k+1} \mathcal{G}(U(a_j), V(a_j))^2 \leq 4\epsilon \int_{a_k}^{a_{k+1}} |D_\tau U|^2 + |D_\tau V|^2 + \frac{4}{\epsilon} \int_{a_k}^{a_{k+1}} \mathcal{G}(U, V)^2 = 4\epsilon K^2$$

. This gives ($\frac{\epsilon}{\lambda} = \frac{1}{l}$)

$$\begin{aligned} \int_{\partial[0, \lambda]^2} |D_\tau \phi_k|^2 &= \frac{1}{l} \left(\int_{a_k}^{a_{k+1}} |D_\tau U|^2 + |D_\tau V|^2 \right) + \sum_{j=k}^{k+1} \frac{\mathcal{G}(U(a_j), V(a_j))^2}{\lambda} \\ &\leq \frac{5}{l} K^2. \end{aligned}$$

Finally we define

$$\widehat{\phi}(x, t) = \phi_k(l(x - a_k), t) \text{ for } (x, t) \in [a_k, a_{k+1}] \times [0, \lambda].$$

Due to lemma A.3 we found

$$\begin{aligned} \text{(A.17)} \quad \int_{[a_k, a_{k+1}] \times [0, \lambda]} |D\widehat{\phi}|^2 &\leq l \int_{[0, \lambda]^2} |D\phi_k|^2 \leq 3Q l \lambda \int_{\partial[0, \lambda]^2} |D_\tau \phi_k|^2 \leq 15Q \lambda K^2 \\ \mathcal{G}(\widehat{\phi}(x, t), U(x))^2 &= \mathcal{G}(\phi_k(y, t), \phi_k(y, 0))^2 \leq 5\pi Q^2 \epsilon K^2 \quad \forall x = a_k + \frac{y}{l}, y \in [0, \lambda]. \end{aligned}$$

Since all sets $[a_k, a_{k+1}] \times [0, \lambda]$ are disjoint we obtain a well defined extension $\widehat{\phi}$ applying the above procedure for every $k = 0, \dots, l-1$. Furthermore adding the estimate (A.17) for $k = 0, \dots, l-1$ we obtain (A.16) proving the lemma. \square

The choice $l = 1$ in lemma A.4 reduces it back to lemma A.3. This corresponds to C. De Lellis proposal in [13] to choose the "harmonic" extension. This is in general not a good idea for the L^∞ -bound. This can be seen in the following example.

Example A.3. Let $F = [0, 1]$, ($\lambda = 1$), $M \in \mathbb{N}$ and $\phi_M(x, 0) = \phi(x, \lambda) = M(\cos(2\pi Mx), \sin(2\pi Mx)) \in W^{1,2}(F, \mathbb{R}^2)$, $\phi(0, t) = \phi(\lambda, t) \equiv (0, 1)$. S. Luckhaus suggests the extension $\widehat{\phi}_L(x, t) = \phi(x, 0)$ for all $t \in [0, \lambda]$ that satisfies $\text{dist}(\widehat{\phi}_L(x, t), \phi(F, 0))^2 = 0$ for all $(x, t) \in F \times [0, \lambda]$. The harmonic extension would be

$$\widehat{\phi}_H(x, t) = \frac{\cosh(2\pi M(t - \frac{1}{2}))}{\cosh(\pi M)} \phi(x, 0);$$

that satisfies now

$$\inf_{x \in F} |\phi_H(x, \frac{1}{2}) - \phi(x, 0)| \geq |\phi(x, 0)| - |\phi_H(x, \frac{1}{2})| = M \left(1 - \frac{1}{\cosh(\pi M)} \right);$$

converging to $+\infty$ as $M \rightarrow \infty$.

Proof of Lemma A.1. (Our presentation is close to the proof presented by R. Moser in [17].)

Part 1: decomposition \mathcal{G} of the sphere using a Fubini-type argument

It is useful to set up some terminology. $\frac{1}{L}\mathbb{Z}^N$ is a square lattice in \mathbb{R}^N decomposing the cube $[-1, 1]^N$ and its boundary $\partial[-1, 1]^N$ into congruent cubes of side length $\frac{1}{L}$ of dimension N and $N-1$. Let \mathcal{F}_k denote the collection of all k -dimensional faces in the decomposition $\partial[-1, 1]^N \cap \frac{1}{L}\mathbb{Z}^N$. We set $\mathcal{G}_k = \{H(F) : F \in \mathcal{F}_k\}$, a collection of k -dimensional faces on the sphere \mathcal{S}^{N-1} . The number of k -dimensional faces $\sharp F_k = \sharp G_k$ is less than $2N$ -times the number of k -dimensional faces in $[-1, 1]^{N-1} \cap \frac{1}{L}\mathbb{Z}^{N-1}$, that is less than $(2L)^{N-1} \binom{N-1}{k}$, in total

$$\text{(A.18)} \quad \sharp \mathcal{F}_k = \sharp G_k \leq N 2^N L^{N-1} \binom{N-1}{k}.$$

claim: Let $f \in L^1(\mathcal{S}^{N-1}, \mathbb{R}_+)$ be given. Then there is a partition of $SO(N)$ into the set $\mathcal{O}^{\text{good}}$ of "good" and the set \mathcal{O}^{bad} of "bad" matrices, defined as follows: $O \in \mathcal{O}^{\text{good}}$ if we have

$$(A.19) \quad \sum_{k=1}^{N-2} \frac{L^{N-1-k}}{\binom{N-1}{k}} \sum_{G \in \mathcal{G}_k} \int_G f(Ox) d\mathcal{H}^k(x) \leq \frac{(N-2)2^N}{\theta w_N} \int_{\mathcal{S}^{N-1}} f d\mathcal{H}^{N-1}$$

and $O \in \mathcal{O}^{\text{bad}}$ if instead

$$(A.20) \quad \sum_{k=1}^{N-2} \frac{L^{N-1-k}}{\binom{N-1}{k}} \sum_{G \in \mathcal{G}_k} \int_G f(Ox) d\mathcal{H}^k(x) > \frac{(N-2)2^N}{\theta w_N} \int_{\mathcal{S}^{N-1}} f d\mathcal{H}^{N-1}.$$

Furthermore one has $\mu(\mathcal{O}^{\text{bad}}) < \theta$, where μ is the Haar measure on $SO(N)$.

This can be seen as follows: To any $x, x_0 \in \mathcal{S}^{N-1}$ there exists $O_0 \in SO(N)$ with $O_0 x_0 = x$ and by the invariance of the Haar measure under group action we have

$$\int_{O \in SO(N)} f(Ox) d\mu(O) = \int_{SO(N)} f(OO_0 x_0) d\mu(O) = \int_{SO(N)} f(Ox_0) d\mu(O).$$

The invariance of the Hausdorff measure under orthogonal transformations gives

$$\int_{\mathcal{S}^{N-1}} f(Ox) d\mathcal{H}^{N-1}(x) = \int_{\mathcal{S}^{N-1}} f(x) d\mathcal{H}^{N-1}(x).$$

Fubini's theorem with $\mu(SO(N)) = 1$ gives

$$\begin{aligned} \int_{\mathcal{S}^{N-1}} f d\mathcal{H}^{N-1} &= \int_{\mathcal{S}^{N-1}} f(Ox) d\mathcal{H}^{N-1}(x) = \int_{SO(N)} \int_{\mathcal{S}^{N-1}} f(Ox) d\mathcal{H}^{N-1}(x) d\mu(O) \\ &= N w_N \int_{SO(N)} f(Ox_0) d\mu(O). \end{aligned}$$

We deduce

$$\begin{aligned} (A.21) \quad & \int_{SO(N)} \sum_{G \in \mathcal{G}_k} \int_G f(Ox) d\mathcal{H}^k(x) d\mu(O) = \sum_{G \in \mathcal{G}_k} \int_{SO(N)} f(Ox_0) d\mu(O) \mathcal{H}^k(G) \\ & \leq 2^N N \binom{N-1}{k} L^{N-1-k} \int_{SO(N)} f(Ox_0) d\mu(O) = \frac{2^N}{w_N} \binom{N-1}{k} L^{N-1-k} \int_{\mathcal{S}^{N-1}} f d\mathcal{H}^{N-1}. \end{aligned}$$

We used (A.18) and $\mathcal{H}^k(G) = \mathcal{H}^k(H(F)) \leq \mathcal{H}^k(F) = L^k$. This implies the claim $\mu(\mathcal{O}^{\text{bad}}) < \theta$ because apply (A.21) for every k and (A.20) for every $O \in \mathcal{O}^{\text{bad}}$ to deduce

$$\begin{aligned} & \frac{\mu(\mathcal{O})}{\theta} \frac{(N-2)2^N}{w_N} \int_{\mathcal{S}^{N-1}} f d\mathcal{H}^{N-1} \\ & < \int_{O \in \mathcal{O}} \sum_{k=1}^{N-2} \frac{L^{N-1-k}}{\binom{N-1}{k}} \sum_{G \in \mathcal{G}_k} \int_G f(Ox) d\mathcal{H}^k(x) d\mu(O) \leq \frac{(N-2)2^N}{w_N} \int_{\mathcal{S}^{N-1}} f d\mathcal{H}^{N-1}; \end{aligned}$$

i.e. $\mu(\mathcal{O}^{\text{bad}}) < \theta$.

Given u, v as assumed, set $\theta = \frac{1}{2}$ and $f_1 = |Du|^2 + |Dv|^2 + \frac{\mathcal{G}(u,v)^2}{\epsilon^2}$, $f_2 = |u|^2 + |v|^2$. The the claim states that if $\mathcal{O}_i^{\text{good}} \cup \mathcal{O}_i^{\text{bad}} = SO(N)$ are the related partition, there exists $O \in \mathcal{O}_1^{\text{good}} \cap \mathcal{O}_2^{\text{good}}$ since $\mu(\mathcal{O}_1^{\text{bad}} \cup \mathcal{O}_2^{\text{bad}}) < 1$. Hence we have for any $k = 1, \dots, N-2$, $G \in \mathcal{G}_k$

$$\begin{aligned} & u \circ O|_G, v \circ O|_G \in W^{1,2}(G, \mathcal{A}_Q(\mathbb{R}^m)) \\ & (u \circ O|_G)|_{G'} = u \circ O|_{G'}, (v \circ O|_G)|_{G'} = v \circ O|_{G'}, \quad \forall G' \in \mathcal{G}_{k-1}, G' \subset \partial G. \end{aligned}$$

We define $U(x) = u(OH(x))$, $V(x) = v(OH(x))$. Due to the choice of O we have that for any $k = 1, \dots, N-2$, $F \in \mathcal{F}_k$

$$\begin{aligned} U|_F, V|_F &\in W^{1,2}(F, \mathcal{A}_Q(\mathbb{R}^m)) \\ (U|_F)|_{F'} &= U|_{F'}, (V|_F)|_{F'} = V|_{F'} \quad \forall F' \in \mathcal{F}_{k-1}, F' \subset \partial F. \end{aligned}$$

Set $\tilde{f}_1 = |DU|^2 + |DV|^2 + \frac{\mathcal{G}(U,V)^2}{\epsilon^2}$ and using remark A.2 we have for any $F \in \mathcal{F}_k$

$$\int_F \tilde{f}_1 d\mathcal{H}^k \leq \int_{G=H(F)} \left(\frac{|x|_2}{|x|_\infty} \right)^{k-1} f_1(Ox) d\mathcal{H}^k(x) \leq N^{\frac{k-1}{2}} \int_G f_1(Ox) d\mathcal{H}^k(x).$$

so that

$$(A.22) \quad \sum_{k=1}^{N-2} \frac{L^{N-1-k}}{N^{\frac{k-1}{2}} \binom{N-1}{k}} \sum_{F \in \mathcal{F}_k} \int_F \tilde{f}_1 d\mathcal{H}^k \leq \frac{(N-2)2^{N+1}}{w_N} K^2.$$

Part 2: extensions of maps that are defined on the boundary of a k -dimensional cube ∂F to its interior

This is covered in the results of lemma A.2 and A.4.

Part 3: recursive construction of ϕ

We define ϕ on $F \times [0, \lambda]$ $\forall F \in \mathcal{F}_1$ using lemma A.4, then recursively on $\{F \times [0, \lambda] : F \in \mathcal{F}_2\}, \{F \times [0, \lambda] : F \in \mathcal{F}_3\}, \dots, \{F \times [0, \lambda] : F \in \mathcal{F}_{N-1}\}$ by lemma A.2. In each step taking advantage of the fact that ϕ had already be defined on the boundary of $F \times [0, \lambda]$, with

$$(A.23) \quad \phi(x, 0) = U(x), \phi(x, \lambda) = V(x) \quad \forall x \in F, F \in \mathcal{F}_k.$$

Now we describe the construction in detail. (D_τ denotes the tangential differential with respect to the domain of integration, i.e. $|D_\tau \phi|^2$ will be the Dirichlet energy with respect to $F \times [0, \lambda]$, $|D_\tau U|^2 + |D_\tau V|^2$ the Dirichlet energy with respect to a face F .) Pick $z \in \mathcal{F}_0$, the set of all vertices, define

$$(A.24) \quad \phi(z, t) \in W^{1,2}(\{z\} \times [0, \lambda], \mathcal{A}_Q(\mathbb{R}^m)) \quad t \mapsto \phi(z, t) \text{ linear between } U(z), V(z).$$

We proceed this way for all $z \in \mathcal{F}_0$: since $z \in \partial F'$ for some $F' \in \mathcal{F}_1$ and $W^{1,2}(F', \mathcal{A}_Q(\mathbb{R}^m)) \subset C^{0, \frac{1}{2}}(F', \mathcal{A}_Q(\mathbb{R}^m))$, $U(z), V(z)$ are defined. Furthermore all $\{z\} \times [0, \lambda]$ are disjoint so ϕ is welldefined on $\bigcup_{z \in \mathcal{F}_0} \{z\} \times [0, \lambda]$.

Pick $F \in \mathcal{F}_1$ then ϕ is already defined on $\partial(F \times [0, \lambda]) = F \times \{0, \lambda\} \cup \partial F \times [0, \lambda]$ taking into account (A.23) and (A.24). We apply lemma A.2 to extend ϕ to $F \times [0, \lambda]$ with the estimates: $\int_{F \times [0, \lambda]} |D_\tau \phi|^2 \leq 15Q \lambda K_F^2$, $\text{dist}(\phi(x, t), U(F) \cup V(F))^2 \leq 15Q^2 \epsilon K_F^2$ with $K_F^2 = \left(\int_F |D_\tau U|^2 + |D_\tau V|^2 + \frac{\mathcal{G}(U,V)^2}{\epsilon^2} d\mathcal{H}^1 \right)$. We can define ϕ for all $F \in \mathcal{F}_1$ since the interior of the sets $F \times [0, \lambda]$ are disjoint. Taking into account (A.22) we found (with $C_1 \leq \frac{2^{N+5}(N-1)^2}{w_N}$)

$$(A.25) \quad \sum_{F \in \mathcal{F}_1} \int_{F \times [0, \lambda]} |D_\tau \phi|^2 \leq 15Q \lambda \sum_{F \in \mathcal{F}_1} K_F^2 \leq C_1 Q \lambda^{3-N} K^2$$

$$(A.26) \quad \text{dist}(\phi(x, t), U(F) \cup V(F))^2 \leq C_1 Q^2 \epsilon \lambda^{2-N} K^2 \quad (x, t) \in \bigcup_{F \in \mathcal{F}_1} F \times [0, \lambda].$$

Pick $F \in \mathcal{F}_2$, then ϕ is defined on $\partial(F \times [0, \lambda]) = F \times \{0, \lambda\} \cup \partial F \times [0, \lambda]$, taking into account (A.23) and the previous step ($\partial F = \bigcup_{i=1}^4 F_i, F_i \in \mathcal{F}_1$). Hence ϕ can be extended to $F \times [0, \lambda]$ using lemma A.2 s.t. $\phi(x, t) \in \{\phi(y, s) : (y, s) \in \partial(F \times [0, \lambda])\}$ and

$$\int_{F \times [0, \lambda]} |D_\tau \phi|^2 \leq \frac{3}{2} \lambda \left(\int_F |D_\tau U|^2 + |D_\tau V|^2 d\mathcal{H}^2 + \sum_{i=1}^4 \int_{F_i \times [0, \lambda]} |D_\tau \phi|^2 \right).$$

As before the interior of the sets $F \times [0, \lambda]$, $F \in \mathcal{F}_2$ are disjoint, so we can proceed this way for all of them and obtain a welldefined ϕ on $\bigcup_{F \in \mathcal{F}_2} F \times [0, \lambda]$. Summing the above estimate for all $F \in \mathcal{F}_2$, taking into account (A.22) and (A.25) we get for some constant C_2 :

$$\sum_{F \in \mathcal{F}_2} \int_{F \times [0, \lambda]} |D_\tau \phi|^2 \leq C_2 Q \lambda^{4-N} K^2.$$

(For a given $F \in \mathcal{F}_k$ we have $\#\{F' \in \mathcal{F}_{k+1} : F \subset \partial F'\} \leq 2(N-1-k)$.)

We use the same method to define ϕ on $\{F \times [0, \lambda] : F \in \mathcal{F}_3\}, \dots, \{F \times [0, \lambda] : F \in \mathcal{F}_{N-1}\}$. Each time we obtain the inequality

$$\sum_{F \in \mathcal{F}_k} \int_{F \times [0, \lambda]} |D_\tau \phi|^2 \leq C_k Q \lambda^{k+2-N} K^2.$$

For $k = N-1$ this is

$$(A.27) \quad \int_{\partial[-1,1]^N \times [0, \lambda]} |D_\tau \phi|^2 \leq C_{N-1} Q \lambda K^2.$$

Applying lemma A.2 does not affect the L^∞ bound, (A.26).

Define $\varphi(x) = \varphi(ry) = \phi(G \circ O^t(y), 1-r) \in W^{1,2}(B_1 \setminus B_{1-\lambda}, \mathcal{A}_Q(\mathbb{R}^m))$, with $r = |x|$, $y = \frac{x}{|x|}$. One checks that ϕ satisfies (A.2). (A.27) combined with remark A.2 gives the energy bound (A.3):

$$\int_{B_1 \setminus B_{1-\lambda}} |D\varphi|^2 \leq 4 \int_{\partial[-1,1]^N \times [0, \lambda]} |D\phi|^2 \leq C Q \lambda K^2.$$

Finally the preserved L^∞ bound (A.26) corresponds with (A.4). \square

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